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Strategy Use and Strategy Choice in Fraction Magnitude Comparison

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We examined, on a trial-by-trial basis, fraction magnitude comparison strategies of adults with more and less mathematical knowledge. College students with high mathematical proficiency used a large variety of strategies that were well tailored to the characteristics of the problems and that were guaranteed to yield correct performance if executed correctly. Students with less mathematical proficiency sometimes used strategies similar to those of the mathematically proficient students, but often used flawed strategies that yielded inaccurate performance. As predicted by overlapping waves theory, increases in accuracy and speed were related to differences in strategy use, strategy choice, and strategy execution. When asked to choose the best strategy from among 3 possibilities—the strategy the student originally used, a correct alternative, and an incorrect alternative—students with lower fraction knowledge rarely switched from an original incorrect strategy to a correct alternative. This finding suggests that use of poor fraction magnitude comparison strategies stems in large part from lack of conceptual understanding of the requirements of effective strategies, rather than difficulty recalling or generating such strategies.

Keywords: fractions, magnitude comparison, strategies, community college

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A common finding across many domains is that with increasing knowledge, people become faster and more accurate. However, the mechanisms behind these improvements are often left unexplored. In accordance with overlapping waves theory (Siegler, 1996), we propose that much of the improvement that occurs with greater experience and knowledge can be traced back to greater use of more efficient strategies, more adaptive strategy choices, and improved execution of strategies. In the experiments that follow, we explore students' strategy use, choice, and execution within an important domain of mathematics: fractions.

Overlapping Waves Theory

One of the main tenets of overlapping waves theory (Siegler, 1996, 2006) is that thinking is variable. At any given time, a person knows multiple ways of solving a problem. The strategy that is chosen depends on characteristics of both the problem and the learner. Thus, not only will different people use different strategies on the same problem, but the same person will use different strategies on similar problems or even on the same problem presented on two separate occasions in the same session (e.g., Siegler, 1995; Siegler & Shrager, 1984). Moreover, choices among strategies are adaptive; people choose among available strategies in ways that yield good combinations of speed and accuracy for the problem presented. For example, kindergartners who are learning simple addition will often retrieve the answer to simple problems such as 2 + 1, but will usually use a more reliable, but slower, backup strategy when solving more difficult problems such as 4 + 3, for which retrieval would be inaccurate (Siegler & Robinson, 1982).

As illustrated in Figure 1, strategy use changes over time. With experience, people rely more heavily on some strategies (Strategy 2, and then Strategy 4, in the figure), whereas less efficient approaches (Strategy 1 in the figure) fade away. At any given time, multiple strategies are in use.

According to overlapping waves theory, four main changes in strategy use result in improved accuracy and speed: Learners discover more advanced strategies, rely more often on the more effective strategies from among the set they already know, choose more adaptively among strategies, and improve their execution of known strategies. This theory has been successfully applied in a number of domains, such as arithmetic (van der Ven, Boom, Kroesbergen, & Leseman, 2012), reading (Lindberg et al., 2011), spelling (Rittle-Johnson & Siegler, 1999), motor skills (Keller,

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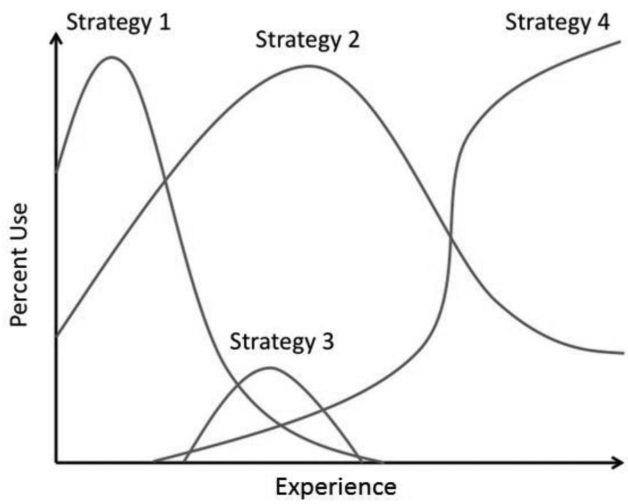

Figure 1. Hypothetical data showing changes in strategy use with increasing experience.

Lamenoise, Testa, Golomer, & Rosey, 2011), infant–mother interactions (Lavelli & Fogel, 2005), and tool use (Chen & Siegler, 2000). In the present research, we applied the theory to a new domain: the understanding of fraction magnitudes.

The Importance of Fraction Magnitude Knowledge

The role of mathematical knowledge in adults' occupational and economic success is well documented. High-paying jobs in medicine, software development and electrical and computer engineering, as well as many moderately paying jobs, such as automotive technician, nurse, and machinist, require substantial mathematical proficiency (Davidson, 2012; McCloskey, 2007; Murnane, Willett, & Levy, 1995; National Mathematics Advisory Panel, 2008a; Rivera-Batiz, 1992). Unfortunately, many adults lack the mathematical proficiency needed for these positions.

The mathematical knowledge needed for reasonably well-paying occupations often centers on fractions and the closely related concepts of ratios and proportions (McCloskey, 2007; Murnane et al., 1995). Fractions also provide a crucial foundation for acquiring more advanced mathematics. Not only are fractions omnipresent in prealgebra, algebra, trigonometry, and other areas of mathematics learned after fractions, equations in these areas are literally meaningless without understanding the fractions involved. For example, without understanding what $\frac{1}{5}$ means, learners cannot understand that "$\frac{1}{5}X = Y$" implies that X is larger than Y, much less that it is exactly 5 times as large. Not surprisingly, given this analysis, a nationally representative sample of 1,000 U.S. algebra teachers rated fractions as one of the two most important failings in their students' preparation for learning algebra—second only to the amorphous category of "word problems" (National Mathematics Advisory Panel, 2008b). Complementing this evidence, knowledge of fractions in fifth grade uniquely predicts overall mathematics achievement in 10th grade in both the United States and the United Kingdom, even after statistically controlling for general intellectual capabilities, such as verbal IQ, nonverbal IQ, and working memory; other mathematical skills, such as whole number arithmetic; and family background variables, such as parental education and income (Siegler et al., 2012).

Fractions also occupy a key role in theories of numerical development such as Siegler, Thompson, and Schneider's (2011) integrated theory of numerical development. This theory notes that fractions allow children to deepen their understanding of numbers beyond the level likely to arise from experience with whole numbers. All whole numbers have unique successors, can be represented as a single symbol, never decrease with multiplication, never increase with division, and so on. Children often assume that these properties of whole numbers are true of other types of numbers as well (Vamvakoussi & Vosniadou, 2004). Experience with fractions allows students to learn that none of these properties are in fact true of numbers in general (though it does not guarantee such learning, see Vamvakoussi & Vosniadou, 2010). Fractions also provide the opportunity to learn the one property that unites all real numbers: They have magnitudes that can be located on number lines and combined arithmetically. Consistent with the hypothesis that understanding fraction magnitudes is central to numerical development, 11- and 13-year-olds' fraction magnitude knowledge is strongly related to mathematics achievement test scores at those ages, even after controlling for fraction arithmetic proficiency (Siegler & Pyke, 2013; Siegler et al., 2011).

Fraction Magnitude Comparison

One way to assess understanding of fraction magnitudes is through magnitude comparison tasks, in which people need to choose the larger of two fractions. Although relatively few studies of fraction magnitude representations have been conducted, a debate has already developed regarding whether adults perform magnitude comparison with fractions by comparing each fraction's integrated magnitude N_1/N_2 (e.g., Schneider & Siegler, 2010) or whether they rely on separate comparisons of numerators or denominators (e.g., Bonato, Fabbri, Umiltá, & Zorzi, 2007). One proposed resolution is that people compare integrated magnitudes when neither numerators nor denominators are equal, but only compare the numerator or the denominator when the other component is equal (Meert, Grégoire, & Noel, 2009, 2010).

This debate, however, ignores the many other ways in which fraction magnitudes can be compared. These include multiplying to obtain common denominators, noticing that the fractions' magnitudes are on opposite sides of one half, and visualizing the fractions as parts of circles or rectangles. Because many strategies are possible, because existing theories propose use of a single strategy for either all problems or for broad classes of problems, and because the overlapping waves theory has not been applied to this domain, fraction magnitude comparison is an ideal task to test predictions of the overlapping waves theory. Describing fraction magnitude comparison at the level of specific strategies used on each trial on specific types of problems will allow for a more nuanced understanding of how people understand and represent fraction magnitudes than would otherwise be possible.

Although a variety of methods have been used to study fraction magnitude representations, including behavioral priming (Meert et al., 2009), fMRI (Ischebeck, Schocke, & Delazer, 2009), single cell recording (Nieder, Freedman, & Miller, 2002), chronometric analyses (DeWolf, Grounds, Bassok, & Holyoak, 2014; Sprute & Temple, 2011), and error analyses

(Schneider & Siegler, 2010), all previous studies have relied on data aggregated over many trials. As such, participants have typically been classified as using either a holistic approach (comparing integrated fraction magnitudes) or a componential approach (comparing numerators only or denominators only). Lost in such analyses is the fact that participants are likely using a number of different strategies, even on highly similar problems (as noted by Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). Past research on whole number arithmetic has shown that although children appeared to be using only one strategy when data were aggregated across trials, trial-by-trial analyses indicated that individual children and adults actually used multiple distinct strategies (LeFevre, Bisanz, et al., 1996; Siegler, 1987).

Goals of the Current Research

We know of only one study that has examined fraction magnitude comparison strategies on a trial-by-trial basis (Faulkenberry & Pierce, 2011). The researchers found evidence for five unique comparison strategies. However, participants in this study were given four example strategies before performing the task, which may have biased the strategies used and reported (the modeled strategies were four of the five approaches used by the participants). In addition, all comparison problems included ½, ⅓, or ⅔, perhaps thereby limiting the range of strategies. In the present study, we examined strategy use without prompting and on a wide range of problems.

In particular, we assessed strategy use on each trial in order to describe the variability in strategies, regularities in choice among strategies, and how both variability and choice varied with mathematical knowledge. We also examined whether the increased accuracy and speed of high-knowledge students, relative to lower knowledge students, could be explained by the four mechanisms hypothesized by overlapping waves theory to produce learning: discovery of more advanced strategies, increased reliance on advanced strategies that were already used to some extent, improved strategy choice, and improved execution of known strategies. Immediately after each trial, students at either a highly selective university or a nonselective community college provided self-reports of the strategy they used to solve the problem.

Such immediately retrospective self-reports have been shown to be a valid and nonreactive measure of strategy use in other numerical domains, such as whole number addition and multiplication (Campbell & Alberts, 2009; LeFevre, Bisanz, et al., 1996; LeFevre, Sadesky, & Bisanz, 1996; although see Thevenot, Castel, Fanget, & Fayol, 2010, for limitations of the self-report method). Like these previously studied domains, fraction magnitude comparisons fit Ericsson and Simon's (1993) criteria for types of tasks and situations in which people usually can provide valid self-reports of strategy use: Problem solutions take enough time to produce symbols in working memory (more than 1 s) and the report is given before the working memory symbols are lost (about 10 s).

Following overlapping waves theory, our first hypothesis was that students would use varied fraction magnitude comparison strategies. In domains as diverse as verb tense (Kuczaj, 1977), recall from memory (Coyle & Bjorklund, 1997), number conservation (Siegler, 1995), false belief (Amsterlaw & Wellman, 2006), and arithmetic (LeFevre, Sadesky, et al., 1996), individual participants use multiple strategies. This variability is present even when identical problems are presented at different times within a single session (Siegler & McGilly, 1989; Siegler & Shrager, 1984). We expected similar variability with adults' fraction magnitude comparison strategies.

Our second hypothesis was that strategy choices would be adaptive, in the sense that each strategy would be advantageous on the type of problem on which it was used most often. To test this hypothesis, we created a wide variety of comparison problems, some of which were designed to be solved easily using a particular strategy (e.g., halves referencing for comparisons involving a fraction larger than one half and a fraction smaller than one half). Past studies have found that strategy choices are adaptive in numerous situations even when people possess little knowledge of a topic (Adolph, Bertenthal, Boker, Goldfield, & Gibson 1997; Lavelli & Fogel, 2005; Lemaire & Siegler, 1995).

Our third hypothesis was that college students with greater mathematical knowledge would not only be more accurate and faster at fraction comparisons, but that this superiority could be explained by the four mechanisms described by overlapping waves theory: discovery of more advanced strategies, increased reliance on known advanced strategies, improved strategy choice, and improved execution of known strategies. Most studies involving overlapping waves theory examine differences among age groups, or changes within an age group with experience, during childhood; this study examines differences among adults who vary in their mathematical competence, comparing strategy use of students from a highly selective university to strategy use of students from a nonselective community college.

Our fourth hypothesis was that suboptimal fraction magnitude comparison strategies are primarily related to a lack of conceptual knowledge about fractions, rather than to a failure to recall superior strategies. If this is the case, adults should continue to use suboptimal strategies even when they are presented superior ones and asked to choose between them and inferior approaches. Adults' conceptual knowledge of fractions has not been given much attention, but data from middle school children (Hecht & Vagi, 2010; Siegler & Pyke, 2013) suggest that unless conceptual knowledge of fractions improves considerably between middle school and adulthood, many adults will be unable to accurately evaluate the usefulness of strategies that are presented to them.

Overview of the Experiments

In Experiment 1, undergraduates at a highly selective university, one in which students tend to be especially proficient in mathematics, were asked to solve a variety of fraction magnitude comparison problems and report their strategy on each trial. Variations in strategy use on different types of problems were used to infer aspects of the strategy choice process. In Experiment 2, students at a nonselective community college, in which students were presumed to be much lower in mathematics proficiency, performed the same task, which allowed identification of differences in their strategy use and that of students at the highly selective university. Experiment 3 examined whether community college students would recognize superior fraction comparison strategies when they were presented. The purpose of this experiment was to determine whether choices of inadequate strategies stemmed from memory failure—being unable to remember an effective strategy for solving each problem but recognizing the superiority of such a strategy when it was presented—or whether use of inadequate strategies stemmed from limited conceptual understanding of fraction comparison, which would be evident if participants did not prefer superior strategies over their own weak strategies, even when they were presented both and asked to choose between them.

Experiment 1

Method

Participants. The participants were 19 undergraduate students (11 female; median age = 20 years) enrolled in introductory psychology courses at a highly selective university in Pennsylvania. The mean self-reported math SAT score was 720 (with a range of 620 to 800). The SAT is a standardized test widely used for college admission in the United States; possible scores range from 200 to 800. This average score is almost identical to the university's mean math SAT score of 718 for incoming freshman during the year the study was conducted, suggesting that self-reports were accurate.

Tasks.

Magnitude comparison. Participants were presented eight types of fraction magnitude comparison problems (see Table 1). Six instances of each type of problem were presented—three with denominators from 2 to 9 inclusive, and three with denominators from 11 to 19 inclusive, resulting in a total of 48 items. All fraction magnitudes were between 0 and 1. The first six types of problems in Table 1 were designed to elicit specific strategies, whereas the large-distance estimation and small-distance estimation problems could be solved in multiple ways. Table S1 of the online supplemental materials provides more information about the particular problems.

Table 1
Examples of Each Problem Type on the Magnitude Comparison Task

Problem type	Larger fraction	Smaller fraction
Equal denominator	3/7	2/7
	13/17	9/17
Equal numerator	3/4	3/5
	2/13	2/17
Larger numerator and smaller denominator	3/7	2/9
	3/11	2/15
Halves reference	2/3	3/7
	11/16	6/13
Multiply for common denominator	2/3	5/9
	3/7	5/14
Multiply for common numerator	4/7	2/9
	8/19	4/15
Large-distance estimation (more than .30 apart)	4/9	1/8
	9/19	2/17
Small-distance estimation (less than .30 apart)	3/4	5/9
	5/12	6/19

The two fractions being compared on each trial were shown on the left and right sides of a computer monitor. Participants were told to press "a" if the fraction on the left was larger and "l" if the fraction on the right was larger. The larger fraction appeared on each side of the screen on half of the trials. After participants indicated the larger fraction, they were asked to explain aloud the strategy through which they chose it, with the problem remaining on the screen while they did so. This information was coded later, based on audio recordings. The stimuli were shown in a stratified random order, such that one item from each of the eight problem types was shown before a second problem of any type was shown.

Number line estimation. To provide convergent validation of the students' fraction knowledge, participants were presented 20 number line estimation problems. On each trial, participants saw a line with "0" at the left end and "5" at the right end, and a fraction that differed on each trial above the line. Participants indicated with a hatch mark where the fraction would go on the line. Four problems were chosen from each fifth of the number line: 1/19, 3/13, 4/7, 8/11, 7/5, 13/9, 14/9, 12/7, 13/6, 19/8, 8/3, 14/4, 13/4, 10/3, 17/5, 7/2, 17/4, 13/3, 9/2, and 19/4.

Procedure. Half of the participants completed the number line estimation task first, and half completed the magnitude comparison task first.

Results

Magnitude comparison.

Accuracy and solution times. The mathematically proficient participants in this experiment correctly solved 96% (*SD* = 4) of the fraction comparison problems and took an average of 5.95 s (*SD* = 2.34) to do so. The average accuracy and solution time for each of the eight problem types is shown in Table S2 of the online supplemental materials.

Strategies. The second author coded participants' strategies based on audio recordings of each trial. The first author also coded the strategies for a random one third of participants; agreement was 90%, so the original codings were used for all analyses. Nineteen distinct strategies were identified (Table 2).

Number of strategies. Individual participants used an average of 11.21 (*SD* = 1.9) different strategies across the 48 problems. Every participant used between six and 15 approaches.

Strategy choices. To test our hypothesis that participants would adapt their strategy use to the demands of the problem, we examined percent use on each type of problem for the 10 strategies that were used on at least 3% of all trials. These 10 strategies included six that were each hypothesized to be the most frequent strategy for one of the six types of problems (the leftmost six columns of Table 3), and four other strategies: general magnitude reference, choosing the fraction with the smaller difference between numerator and denominator, converting fractions to decimals or percentages, and guessing. Strategies used on fewer than 3% of trials, along with procedures that could not be classified, were categorized as "other" for this analysis. The bolded percentages are located at the intersection of a problem type and the strategy expected to be most frequent on it.

Table 2
Fraction Magnitude Comparison Strategies

General strategy group	Strategies included	Strategy description
Logical necessity: Strategies yield correct answers on all applicable problems	Equal denominators*	If both fractions have equal denominators, the fraction with the larger numerator is larger.
	Equal numerators*	If both fractions have equal numerators, the fraction with the smaller denominator is larger.
	Larger numerator and smaller denominator*	The larger fraction has a larger numerator and a smaller denominator than the smaller fraction.
Intermediate steps: Strategies yield correct answer on all applicable problems if intermediate steps are executed correctly	Multiply for a common denominator*	Multiply one fraction by 1 (in the form of fraction a/a) in order to get common denominators.
	Multiply for a common numerator*	Multiply one fraction by 1 (in the form of fraction a/a) in order to get common numerators.
	Halves reference*	The larger fraction is greater than ½ and the smaller fraction is smaller than ½.
	Convert to decimal/percent*	Convert the fraction into its decimal or percent form and compare the values in those forms.
	General magnitude reference*	Compare one or both fractions to a nearby known magnitude, such as 0, ½, or 1.
	Numerator goes into the denominator fewer times	Divide each denominator by the numerator; the larger fraction yields the smaller answer.
Usually correct: Strategies that yield better than chance results, but do not guarantee correct answers.	Denominators close, pick larger numerator*	When the denominators of the two fractions are very close, the fraction with the larger numerator is greater.
	Numerators close, pick smaller denominator	When the numerators of the two fractions are very close, the fraction with the smaller denominator is greater.
	Difference between numerator and denominator within each fraction is smaller*	The difference between the numerator and denominator of the larger fraction is smaller than the difference between the numerator and denominator of the smaller fraction.
	Visualization*	Using a pie, pizza, or other visual representation of a fraction in order to compare magnitudes.
	Smaller denominator*	The larger fraction has a smaller denominator.
	Transformation	Transform one or both of the fractions, (often rounding one fraction to a known magnitude).
	Larger numerator	*The larger fraction has a larger numerator*
Questionable: Strategies not guaranteed to yield above chance performance	Guess	Stated that s/he guessed.
	Intuition	Just knew that one was larger.
	Larger numerator and denominator*	The larger fraction has a larger numerator and denominator.
	Smaller numerator and denominator*	The larger fraction has a smaller numerator and denominator.
	*Larger denominator**	*The larger fraction has a larger denominator.*
	Smaller numerator	*The larger fraction has a smaller numerator.*
	Larger fraction cannot be reduced	*The fraction is larger because it cannot be reduced.*
	Familiarity	*The fraction is larger because it is more familiar.*
	*Numerator goes into the denominator more times**	*Divide each denominator by the numerator, the larger fraction yields the larger answer.*
	Difference between numerator and denominator within each fraction is larger	*The difference between the numerator and denominator of the larger fraction is larger than the difference between the numerator and denominator of the smaller fraction.*
	*Larger remainder when denominator is divided by numerator**	*The larger fraction has a larger remainder when divided by the numerator (e.g.* $4/10$*, remainder 2, is larger than* $4/9$*, remainder 1).*
	Smaller remainder when denominator is divided by numerator	*The larger fraction has a smaller remainder when divided by the numerator (e.g.* $4/10$*, remainder 2, is smaller than* $4/9$*, remainder 1).*

Note. Strategies in italics were not used in Experiment 1. Strategies marked with an asterisk were coded for in Experiment 3.

In all six cases in which a particular strategy was expected to be most frequent, the most frequent strategy was the predicted strategy. On four of the six, that strategy was used on an absolute majority of trials. Almost all other strategies that were used were also appropriate; for example, general magnitude reference is appropriate on all problems and was reported fairly frequently on several of them.

Another phenomenon that stands out in Table 3 is that participants were more likely to rely on the equal denominator strategy when denominators were equal, than on the equal numerator strategy when numerators were equal. Not only were they more likely to use the equal denominator strategy than the equal numerator strategy—they were also more likely to multiply to produce common denominators than to multiply to produce common numerators.

To further examine participants' strategy choices, we grouped the full set of strategies into the four categories shown in Table 2: (a) *logical necessity strategies*—strategies that would yield perfect performance relying only on information given in the problem; (b) *intermediate steps strategies*—strategies that would yield perfect

Table 3
Percent Use of Strategies by Problem Type by Students at the Highly Selective University (Experiment 1)

Problem type	Equal denom	Equal num	Larger num and smaller denom	Halves reference	Multiply for common denom	Multiply for common num	General magnitude reference	Convert to a decimal	Difference between num and denom	Guess	Other
	Strategy										
Equal denom	**95**	0	0	0	0	0	0	0	5	0	0
Equal num	0	**68**	0	0	8	0	2	12	10	0	1
Larger num and smaller denom	0	0	**45**	0	16	0	9	8	14	1	8
Halves reference	0	0	0	**61**	9	0	0	6	5	4	14
Multiply for common denom	0	0	0	1	**65**	0	14	7	4	4	6
Multiply for common num	0	0	0	2	10	**25**	13	16	9	11	15
Large-distance estimation	0	0	0	0	5	2	37	8	22	3	24
Small-distance estimation	0	0	0	2	13	0	32	11	12	4	26
M	12	8	6	8	16	3	13	8	10	3	12

Note. The bolded numbers indicate the intersection of a problem type and the expected strategy. "Other" includes all strategies that were used on less than 3% of the trials. num = numerator; denom = denominator.

performance if a simple arithmetic transformation of the information in the problem was performed correctly, such as multiplying by N/N to create common denominators; (c) *usually correct strategies*—strategies that would usually, but not always, yield correct answers if executed correctly, such as choosing the fraction with the smaller denominator[1]; and (d) *questionable strategies*—strategies not guaranteed to yield above chance performance, such as noting that both numbers in the chosen fraction were larger. Procedures that were uninterpretable (<1% of trials) were excluded from this categorization.

As shown on the right of Figure 2, participants used logical necessity strategies on 26% ($SD = 6$) of trials, intermediate steps strategies on 51% ($SD = 17$), usually correct strategies on 16% ($SD = 11$), and questionable strategies on 7% ($SD = 7$). All participants used logical necessity and intermediate steps strategies at least once, 89% used usually correct strategies, and 74% used questionable strategies.

Accuracy of strategies. Mean percent correct when using each type of strategy is shown on the right side of Figure 3. Consistent with the categorization, correct answers were produced more often by logical necessity strategies (99% correct, $SD = 2$), intermediate steps strategies (97%, $SD = 6$), and usually correct strategies (95%, $SD = 11$) than by questionable strategies (78%, $SD = 27$), $F(3, 36) = 4.54$, $p = .008$, $\eta_p^2 = .28$. Accuracy on the logical necessity strategies was nearly perfect (a total of one incorrect answer). Intermediate steps strategies also yielded very accurate answers, but occasional computational errors led to some incorrect comparisons. Accuracy when using usually correct strategies approximated what would be expected when such a strategy was applied to the comparison problems presented. For example, the most common usually correct strategy (10% of total trials) was to pick the fraction with the smaller difference between the numerator and the denominator. This strategy yielded correct answers on 95% of items in the problem set, quite close to the 100% correct that emerged when participants used the strategy.

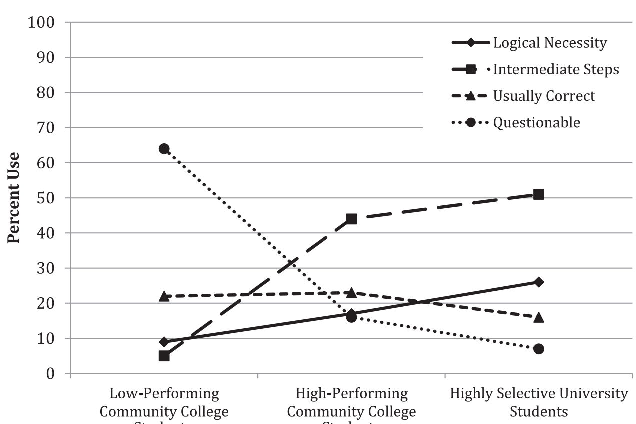

Figure 2. Percent strategy use by students in Experiments 1 and 2. Error bars are standard errors.

In contrast, these mathematically proficient students tended to be far more accurate when they used questionable strategies than might have been expected. The two most frequent questionable strategies were guessing and intuition (3% and 2% of trials, respectively). Participants were correct on 79% of trials on which they reported guessing and 98% of trials on which they reported relying on intuition. Both accuracy rates were much higher than the 50% that would be expected by chance, $t(8) = 4.09$, $p = .003$, $d = 1.36$, $t(8) = 30.50$, $p < .001$, $d = 10.00$, respectively. Thus, these mathematically proficient students' guesses were educated, and their intuitions almost perfect.

Number line estimation. Accuracy of number line estimation was indexed by percent absolute error (PAE), defined as PAE = (|Participant's Answer – Correct Answer|)/Numerical Range × 100. For example, if a participant was asked to locate 5/2 on a 0-to-5 number line, and marked the location corresponding to 3/2, PAE would be 20% ((|1.5 – 2.5|)/5 × 100). PAE varies inversely with accuracy: the higher the PAE, the less accurate the estimate.

Mean PAE was very low, 5% ($SD = 2$), which, like the magnitude comparison data, indicated that these students' fraction magnitude representations were very accurate. Individual participants' percent correct magnitude comparisons correlated,

[1] All usually correct strategies would lead to greater than chance performance on the problems presented or on problems created by randomly choosing numerators and denominators from the same set of numbers.

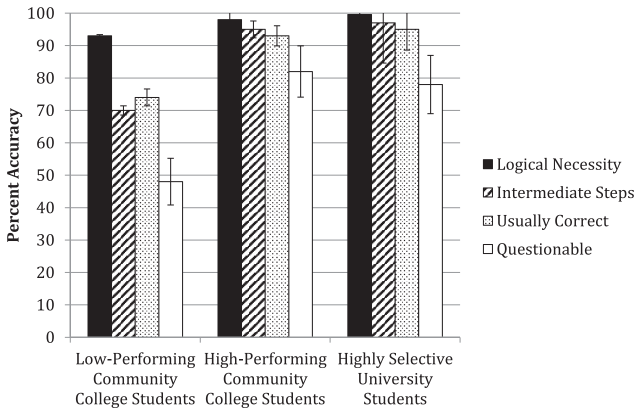

Figure 3. Percent correct when using different types of strategies in Experiments 1 and 2. Error bars are standard errors.

$r(17) = -0.55$, with their number line PAEs; the negative correlation reflected high knowledge, implying high magnitude comparison accuracy but low PAEs. The substantial correlation between number line estimation and magnitude comparison accuracy indicates that even within this highly selective sample, there were meaningful individual differences in the accuracy of fraction magnitude representations.

Discussion

The mathematically proficient students in Experiment 1 demonstrated very accurate fraction magnitude representations, correctly answering 96% of magnitude comparison items and generating fraction number line estimates as accurate as the estimates for whole numbers in the 0-to-1,000 range of students at the same university in Siegler and Opfer (2003). The students were consistently accurate and quick to solve the fraction comparison problems. Consistent with our main hypotheses, they relied on a wide range of fraction magnitude comparison strategies, using each of 10 approaches on at least 3% of trials, with individual participants on average using 11 distinct strategies. Moreover, they tailored their strategy choices to the demands of the specific fraction comparison problems. On all six types of problems with clear strategy predictions, the predicted effective strategy was the modal approach.

Several problem characteristics were related to the frequency with which students used the strategy that seemed maximally efficient. One variable was whether the fractions being compared differed in both numerator and denominator, or in only one of these. When the two numerators or the two denominators were equal, students were especially likely to choose the strategy that seemed optimal, basing judgments on the remaining component (e.g., on $^{13}/_{17}$ vs. $^{9}/_{17}$, comparing 13 and 9). These strategies had the advantage of simplicity and ease of execution when they were applicable.

Another influence on strategy choices was whether it was the numerator or the denominator that was initially equal, or that could become equal through multiplication. Students compared numerators when denominators were equal considerably more often than they compared denominators when numerators were equal (95% vs. 68% of trials). Similarly, on the two problem types on which multiplying one fraction could create equal numerators or denominators, students more often created equal denominators than equal numerators (65% vs. 25% of trials). The difference in strategy choices on these formally equivalent problems might be attributable to students preferring to compare variables that vary positively with the outcome (greater numerator implies greater magnitude); to their viewing fractions with equal denominators as more comparable than ones with equal numerators, perhaps because fractions with equal denominators can be added and subtracted without any further transformation but ones with equal numerators cannot; or to their having more experience with attending to whether denominators are equal and with transforming problems to have equal denominators than with doing the same with numerators (again because of fraction addition and subtraction experience).

A third variable that influenced strategy choices was whether the fractions were on opposite sides of one half. When this was the case, students usually used the fact to explain their answers (61% of trials). One half seems to be a prominent landmark in mental representations of fractions (e.g., Siegler & Thompson, 2014; Spinillo & Bryant, 1991), with people coding whether fractions between 0 and 1 are larger than one half. The same pattern is present with whole numbers, in which magnitudes are coded relative to the midpoint of the range, such as whether numbers between 0 and 100 are greater than 50 (Ashcraft & Moore, 2012).

One further finding of interest concerned these mathematically proficient students' rather accurate use of questionable strategies. Even when these students used strategies such as guessing and intuition that would have yielded 50% correct answers on randomly chosen problems, they generated 78% correct comparisons. This finding suggests that they had more knowledge about fraction magnitudes than they were able to articulate.

To examine how strategy use and choice vary with mathematical proficiency, we presented the same problems to community college students in Experiment 2.

Experiment 2

Method

Participants. Participants were 19 students (15 female; median age = 21 years) who were recruited from introductory psychology and history courses at a nonselective community college in Pennsylvania. Eleven participants reported their SAT math score; their reported mean was 503, with a range of 420 to 720. Only one participant reported a score over 540; thus, there was almost no overlap with the scores of the students from the highly selective university.

Design and procedure. The problems and procedure were those in Experiment 1.

Results

Magnitude comparison.

Accuracy. The community college students correctly answered 74% (SD = 25) of the problems, Mdn = 88%. Accuracy varied widely, with 10 students being quite accurate (mean percent correct = 93%, range = 88% to 98%), and the other nine being quite inaccurate (mean percent correct = 52%, range = 15% to

79%). For the rest of the analyses, we used this difference in magnitude comparison performance to split the students into high-performing and low-performing subgroups. Note that the terms *high-performing* and *low-performing* refer only to their accuracy on the fraction magnitude comparison task, not their overall math or academic achievement. The high-performing students' average accuracy of 93% was just slightly lower than the 96% for the university students in Experiment 1, $t(27) = 2.06$, $p = .049$, $d = 0.80$. The low-performing students' mean accuracy of 52% was lower than those of both the high-performing students, $t(17) = 6.67$, $p < .001$, $d = 3.60$, and students at the highly selective university, $t(26) = 9.83$, $p < .001$, $d = 3.88$. Accuracy rates across the eight problem types are shown in Table S2 of the online supplemental materials.

Solution times. Participants' solution times exceeded 45 s on less than 1% of trials; therefore, outlying values were trimmed to 45 s. Relative to the mathematically proficient students in Experiment 1, solution times were longer among both the high-performing community college students ($M = 5.95$ s vs. 9.33 s, $SD = 3.70$), $t(27) = 3.01$, $p = .006$, $d = 1.12$, and the low-performing ones ($M = 5.95$ s vs. 10.91 s, $SD = 6.14$), $t(26) = 3.12$, $p = .004$, $d = 1.17$. Solution times of the high- and low-performing community college students did not differ, $t < 1$. The average solution time for each of the eight problem types is shown in Table S2 of the online supplemental materials.

Strategies. The community college students used several strategies not observed with the students at the highly selective university (shown in italics in Table 2). One of the novel strategies fit into the usually correct category (choosing the fraction with the larger numerator), but the rest fell into the questionable category. Furthermore, some of the strategies showed no understanding of fraction magnitudes, such as deciding that the fraction that could not be reduced was larger. Again, one third of the participants were double-coded. Agreement was 93%, so the original codings were used for all analyses.

Number of strategies. Individual community college students used an average of 9.95 ($SD = 3.47$) different strategies (range = 4 to 16). The number of different strategies used by the high-performing students ($M = 11.70$, $SD = 2.31$) did not differ from that of the mathematically proficient students in Experiment 1 ($M = 11.21$, $SD = 1.90$), $t < 1$. However, the low-performing community college students used fewer strategies ($M = 8.00$, $SD = 3.61$) than the high-performing students, $t(17) = 2.69$, $p = .015$, $d = 1.24$, and the mathematically proficient students from Experiment 1, $t(26) = 3.09$, $p = .005$, $d = 1.25$.

Strategy choices. Tables 4 and 5 show the strategies used by the high- and low-performing community college students. Like the students at the highly selective university in Experiment 1, the high-performing community college students always used the hypothesized optimal strategy more often than any other strategy on the relevant problems. However, the high-performing community college students were less adaptive in their strategy choices than the students at the highly selective university. For example, the university students used halves referencing on 61% of the halves reference problems, but the high performers at the community college only used it on 25% of such problems. Similarly, the high-performing community college students were even less likely to rely on equal numerators than the students at the highly selective university. Even when the problem encouraged multiplying for a common numerator, the higher performers among the community college students only did so on 3% of problems.

The low-performing community college students showed a very different pattern of strategy use from either of the other groups. On four of the six types of problems with strategy predictions, they relied on the hypothesized strategy on fewer than 2% of trials. The only exceptions were the two types of trials that could be solved by comparing numerators only or denominators only, and even on those trials, the predicted strategy was used on less than half of trials. Fully 84% of the strategies of students in the low performing subgroup were either guesses or strategies that were used on less than 3% of the trials by the mathematically sophisticated students in Experiment 1 and therefore placed in the "other" category in Table 5. The most common strategies in the "other" category for the low-performing community college students were choosing: the fraction with the smaller denominator (13% of trials), the larger denominator (6%), the larger numerator and denominator (8%), the smaller numerator and denominator (9%), and the fraction that yielded the larger remainder when the denominator was divided by the numerator (6%). None of these strategies yielded highly accurate performance.

Table 4
Percent Use of Strategies by Problem Type for High-Performing Community College Students (Experiment 2)

	Strategy										
Problem type	Equal denom	Equal num	Larger num and smaller denom	Halves reference	Multiply for common denom	Multiply for common num	General magnitude reference	Convert to a decimal	Difference between num and denom	Guess	Other
Equal denom	**70**	0	0	0	0	0	10	0	10	2	8
Equal num	0	**42**	0	0	7	0	12	10	15	2	13
Larger num and smaller denom	0	0	**27**	0	10	0	20	0	18	8	17
Halves reference	0	0	0	**25**	10	0	10	5	10	20	20
Multiply for common denom	0	0	0	2	**53**	0	12	2	13	5	13
Multiply for common num	0	0	0	0	12	**3**	27	10	12	13	23
Large-distance estimation	0	0	0	3	2	0	47	3	22	3	20
Small-distance estimation	0	0	0	0	12	0	30	8	13	7	30
M	9	5	3	4	13	0	21	5	14	8	18

Note. The bolded numbers indicate the intersection of a problem type and the expected strategy. "Other" includes all strategies that were used on less than 3% of the trials in Experiment 1. num = numerator; denom = denominator.

Table 5
Percent Use of Strategies by Problem Type for Low-Performing Community College Students (Experiment 2)

	Strategy										
Problem type	Equal denom	Equal num	Larger num and smaller denom	Halves reference	Multiply for common denom	Multiply for common num	General magnitude reference	Convert to a decimal	Difference between num and denom	Guess	Other
Equal denom	**43**	0	0	0	0	0	7	0	0	15	35
Equal num	0	**28**	0	0	0	0	2	6	4	13	48
Larger num and smaller denom	0	0	**2**	0	0	0	0	2	6	26	65
Halves reference	0	0	0	**0**	0	0	0	0	2	19	80
Multiply for common denom	0	0	0	0	**0**	0	0	2	2	24	72
Multiply for common num	0	0	0	0	0	**0**	0	0	4	22	74
Large-distance estimation	0	0	0	0	0	0	6	0	4	13	78
Small-distance estimation	0	0	0	0	4	0	0	6	4	17	70
M	5	3	0	0	0	0	2	2	3	19	65

Note. The bolded numbers indicate the intersection of a problem type and the expected strategy. "Other" includes all strategies that were used on less than 3% of the trials in Experiment 1. num = numerator; denom = denominator.

We next divided the strategies into the same four types as in Experiment 1. The community college students used logical necessity strategies on 13% ($SD = 9$) of trials, intermediate steps strategies on 25% ($SD = 28$), usually correct strategies on 22% ($SD = 21$), and questionable strategies on 39% ($SD = 35$). As shown in Figure 2, the distribution of strategy categories of the high-performing students' was similar to that of the university students in Experiment 1. Consistent with this summary, a 2 (high-performing community college vs. highly selective university) × 4 (strategy type) ANOVA showed no interaction between strategy type and participant group, $F(3, 81) = 1.95$, $p = .13$, $\eta_p^2 = .07$.

In contrast, the distribution of strategy categories of the low-performing community college students differed greatly from that of both the high-performing community college students, $F(3, 51) = 11.70$, $p < .001$, $\eta_p^2 = .41$, and the students at the highly selective university, $F(3, 78) = 41.49$, $p < .001$, $\eta_p^2 = .62$. Compared with the high-performing community college students, the low-performing students more often used questionable strategies, 64% versus 16%, $t(17) = 4.25$, $p = .001$, $d = 1.94$, and less often used both logical necessity strategies, 9% versus 17%, $t(17) = 2.13$, $p = .048$, $d = 0.94$, and intermediate steps strategies, 5% versus 44%, $t(17) = 4.16$, $p = .001$, $d = 1.92$. There were no differences in their use of usually correct strategies, 22% versus 23%, $t < 1$.

Accuracy. The high-performing community college students' pattern of accuracy across strategy types was very similar to that of the highly selective university students, all $ps > .27$ (see Figure 3). The low-performing community college students were as accurate as the high-performing community college students when they used logically correct strategies, (93% vs. 98%), $t < 1$. However, they were less accurate when using intermediate steps strategies (70% vs. 95%), $t(13) = 2.65$, $p = .02$, $d = 1.48$, usually correct strategies (74% vs. 93%), $t(16) = 2.85$, $p = .012$, $d = 1.36$, and questionable strategies (48% vs. 82%), $t(16) = 2.82$, $p = .012$, $d = 1.33$. Thus, the low-performing students' inaccuracy occurred because of a combination of frequent use of questionable strategies and incorrect execution of potentially effective strategies.

Number line estimation. Although classification of the community college students as high or low performing was based on their magnitude comparison performance, the high-performing subgroup was much more accurate at placing fractions on the number line (PAE = 10%, $SD = 4$) than the low-performing group (PAE = 27%, $SD = 8$), $t(17) = 5.72$, $p < .001$, $d = 2.70$. This confirms that the high-performing group was more knowledgeable about fractions in general, not just on the comparison task. Estimates of students at the highly selective university (PAE = 5%, $SD = 2$) were more accurate than those of both the high-performing community college students, $t(27) = 4.27$, $p < .001$, $d = 1.58$, and the low-performing group, $t(26) = 11.23$, $p < .001$, $d = 4.33$. As with the university students, there was a substantial correlation among the community college students between accuracy on the magnitude comparison and number line estimation tasks, $r(17) = -.67$, $p = .002$.

Discussion

The range of performance across the community college students was striking. Accuracy on the fraction comparison task of the high-performing community college students was similar to that of the students at the highly selective university, although the community college students were much slower and less likely to match their strategy to the type of problem. In particular, they were less likely to base comparisons on the fractions being on opposite sides of one half or on one fraction having both a larger numerator and a smaller denominator.

In contrast, the low-performing community college students relied on fewer strategies, used questionable strategies on a majority of trials, and often failed to choose strategies on the basis of opportunities afforded by the specific problem. Unlike in many other domains in which low-performing participants make adaptive strategy choices, these participants did so only minimally. The low-performing students were also much more likely to rely on componential strategies, such as choosing the fraction on the basis of denominator size, even when both numerators and denominators differed for the fractions being compared.

The results of Experiment 2 led us to ask why the low-performing community college students used questionable strategies so often. Did these questionable strategies reflect a failure to remember or generate correct alternatives, or was it related to these

Table 6
Examples of Alternative Strategies Presented in Experiment 3

Problem	Correct alternative	Incorrect alternative
$2/9$ vs. $1/4$	Multiply $1/4$ by 2 to get $2/8$, which has the same numerator as $2/9$. Since we now have equal numerators, the fraction with smaller denominator, 8, is larger. So $1/4$ is larger.	Since $2/9$ has a bigger numerator and denominator, it is larger than $1/4$.
$3/7$ vs. $2/3$	$2/3$ is larger than $1/2$ and $3/7$ is smaller than $1/2$ so $2/3$ is larger.	Since $3/7$ has a bigger denominator than $2/3$, $3/7$ is larger.
$7/12$ vs. $18/19$	The difference between 18 and 19 is smaller than the difference between 7 and 12, so $18/19$ is larger.	Since $7/12$ has a smaller denominator, $7/12$ is larger than $18/19$.
$4/7$ vs. $7/8$	If we multiply to get a common denominator, 56, we will have $7^*7 = 49$ and $4^*8 = 32$, so $7/8$ is larger.	7 divided by 4 is larger than 8 divided by 7, so $4/7$ is larger.
$1/8$ vs. $4/9$	$4/9$ is closer to $1/2$ and $1/8$ is closer to 0, so $4/9$ is larger.	Since $1/8$ has a smaller numerator and denominator than $4/9$, $1/8$ is larger.

students not recognizing the difference between correct and incorrect procedures? To address this question, Experiment 3 examined whether community college students would choose more effective strategies if such strategies were explicitly presented to them as options. If failure to recall effective strategies was the source of the problem, this procedure would lead to consistent choices of effective strategies. On the other hand, if the problem was failing to understand the difference between effective and ineffective strategies, the procedure would not result in such improved choices.

Experiment 3

Method

Participants. Participants were 26 students (16 female; median age = 22) recruited from introductory math courses at two nonselective community colleges in Pennsylvania and California. The mean SAT math score for the five participants who reported their scores was 513. The scores ranged from 300 to 650.

Design. This experiment used a two-phase within-participants design. In the first phase, students were presented four items from each of the eight magnitude comparison problem types used in Experiments 1 and 2. In the second phase, participants were presented the same 32 magnitude comparison problems and asked which of three alternatives would be best for solving the problem. These alternatives were (a) their original strategy, (b) a correct alternative, and (c) an incorrect alternative. To generate these alternatives, we created four alternative strategies for each problem: a primary correct and incorrect strategy, and two backup strategies (one correct, one incorrect). If the student's original strategy was the same as the primary strategy, then the relevant backup strategy (correct or incorrect) was presented (Table 6 provides examples of correct and incorrect alternatives; the full list of primary and backup strategies for each problem is presented in Table S3 of the online supplemental materials). All of the primary correct alternatives were logical necessity or intermediate steps strategies; a few of the correct backup strategies were usually correct strategies that would result in the correct answer on the problem.[2] The incorrect alternatives were taken from the questionable and usually correct strategies, and would always result in incorrect answers on the problem on which they were suggested.

Procedure. The first phase of the experiment used the same procedure as in Experiments 1 and 2. Participants were asked to judge which of two fractions was larger, and then to describe their strategy. The experimenter next pressed a key to indicate which strategy the participant used. Possible strategies included the 17 marked with an asterisk in Table 2 (the more common strategies from Experiments 1 and 2). All other strategies were coded as "other."

Immediately after Phase 1, the second phase of the experiment was presented. It involved the same 32 magnitude comparison problems, excluding any trials on which participants said they were guessing or the strategy was classified as "other." Participants were shown the problem and three potential strategies: the participant's original strategy, another strategy that yielded the same choice as the participant's original strategy, and a third strategy that yielded the opposite outcome (the program was individually tailored for each participant based on the experimenter's coding of his or her strategies during Phase 1). Participants were told that one of the strategies might be the same as their original, but that they should not use that as a basis to choose which strategy to pick. Rather, they should choose whichever strategy would best solve the problem. After choosing, participants were asked to explain their strategy choice. As with Experiments 1 and 2, the entire experiment was computerized and presented using E-Prime programing software (Psychology Software Tools, Inc.).

Results

Magnitude comparison.

Accuracy. The community college students correctly answered 83% ($SD = 16$) of the fraction magnitude comparisons, $Mdn = 91\%$. To keep the knowledge of the high-performing and low-performing groups similar across Experiments 2 and 3, we used the same cutoff as in Experiment 2 (85% correct), rather than a median split. Given the higher accuracy in Experiment 3, a median split would have grouped two students with relatively high accuracy (88%) as low-performers. With the 85% cutoff, there were 16 high-performers and 10 low-performers. The high-performing community college students had a mean accuracy of 93% ($SD = 3$, range = 88% to 97%). The low-performing com-

[2] Because the backup strategies were only presented when the student's original strategy was the primary alternative, students only saw the usually correct alternatives when their original strategy was either a logically correct or intermediate steps strategy.

Table 7
Percent Use of Each Type of Strategy During Phase 1 (Experiment 3)

Achievement group	Logical necessity	Intermediate steps	Usually correct	Questionable	Guess/Other
Low-performing	7 (6)	18 (20)	42 (33)	25 (23)	8 (11)
High-performing	14 (9)	48 (23)	26 (17)	6 (6)	6 (7)

Note. Standard deviations are in parentheses.

munity college students had a mean accuracy of 67% (SD = 15, range = 41% to 84%).

Original strategies.

Number of strategies. Individual participants used an average of 7.46 strategies (SD = 2.83, range = 3 to 13) from among the 18 strategies that were coded (17 distinct strategies and "other"). Number of strategies did not differ ($t < 1$) between high-performing students (M = 7.69, SD = 2.85) and low-performing students (M = 7.10, SD = 2.92). Because only 17 strategies were coded for in this experiment (compared with 27 in Experiments 1 and 2), these numbers are not directly comparable across the three experiments.

Strategy choices. As in Experiment 2, the high-performing community college students adjusted their strategy choices to the different types of problems (see Tables S4 and S5 of the online supplemental materials). As with the community college students in Experiment 2, they were less likely to use the halves reference strategy than the students at the highly selective university in Experiment 1 (25% vs. 61% of trials), and almost never used the strategy of multiplying for a common numerator (2% vs. 25% of trials). Also as in Experiment 2, the low-performing students rarely used the strategies that the problems were designed to elicit. The one difference from Experiment 2 was that low-performing participants occasionally multiplied to generate a common denominator, a strategy rarely used by low-performing peers in Experiment 2.

As shown in Table 7, the low-performing community college students were again less likely than their high performing peers to use intermediate steps strategies, 18% versus 48%, $t(24) = 3.48$, $p = .002$, $d = 1.43$, and more likely to use questionable strategies, 25% versus 6%, $t(24) = 3.14$, $p = .004$, $d = 1.30$. The low-performing students were also marginally less likely to use logical necessity strategies, 7% versus 14%, $t(24) = 2.03$, $p = .053$, d = 0.88.

Strategy switching. This experiment's main focus was on participants' choices among three explicitly presented approaches: their original strategy, a different strategy that led to the correct answer, and a different strategy that led to the incorrect answer. On average, students were not presented two (SD = 2.69) of the original 32 trials in the second phase, because they guessed or used a rare strategy on the problem in the first phase.[3]

On most trials, participants chose their original strategy (M = 57%, SD = 20). When they switched, they more often chose the correct alternative (M = 34%, SD = 19) than the incorrect one (M = 9%, SD = 11), $t(25) = 5.23$, $p < .001$, $d = 1.06$. Low-performing students were more likely to switch to an incorrect strategy (M = 15%, SD = 15) than were high-performing students (M = 5%, SD = 7), $t(24) = 2.32$, $p = .029$, $d = 0.93$, and were marginally less likely to choose the same strategy (low-performing, M = 48%, SD = 22; high-performing, M = 62%, SD = 17), $t(24) = 1.80$, $p = .084$, $d = 0.72$. There were no differences in likelihood of switching to a correct strategy, high-performing (M = 33%, SD = 18), low-performing (M = 37%, SD = 22), $t < 1$.

We next examined Phase 2 switches from the perspective of whether the participant's original strategy was classified as logical necessity, intermediate steps, usually correct, or questionable. The high-performing and low-performing community college students showed very different patterns of choices (see Figure 4). To better deal with missing data (missing in the sense that some participants did not use all four types of strategies in Phase 1), we conducted a mixed-effects linear model analysis rather than a repeated-measures ANOVA.

We first examined the likelihood of participants choosing their original strategy from among the three alternatives, depending on the original strategy's category (logical necessity, intermediate steps, usually correct, or questionable) and participants' classification as high or low performers (with original strategy and achievement level nested within participants). For high-performing students, there was a clear relation between the quality of their original strategy and the likelihood that they switched strategies. High-performing participants were most likely to maintain their original choice when it was a logical necessity strategy and least likely to maintain their choice when the original strategy was questionable, $b = -19.94$, $F(1, 38) = 31.96$, $p < .001$. In contrast, the low-performing students' choice to stay with their original strategy was unrelated to the quality of that strategy, $b = -4.63$, $F(1, 20) = 1.23$ $p = .28$. Confirming that the relation between original strategy quality and the likelihood of continuing to choose that strategy differed between the two groups, there was a significant interaction between student group and original strategy for the probability of sticking with the original strategy, $F(1, 58)$ = 6.96, $p = .011$.

A similar pattern emerged for switches to a correct strategy. High-performing students were more likely to switch to the correct strategy when their original strategy was questionable or usually correct, $b = 13.46$, $F(1, 38) = 18.03$, $p < .001$. Low-performing peers showed no relation between likelihood of switching to the correct strategy and quality of their original strategy, $b = 3.37$, $F < 1$. The interaction was marginally significant, $F(1, 58) = 3.19$, $p = .079$.

Finally, we examined switches to the incorrect strategy. High-performing participants tended to switch to the incorrect strategy

[3] For 2 of the 32 problems, programming errors led to the incorrect strategy either being factually incorrect (5/9 has a smaller denominator than 7/8) or a questionable strategy was used to get the correct answer (6/17 is larger than 4/15 because it has a larger numerator and denominator). The following analyses exclude those two problems. The same pattern of results occurs when they are included.

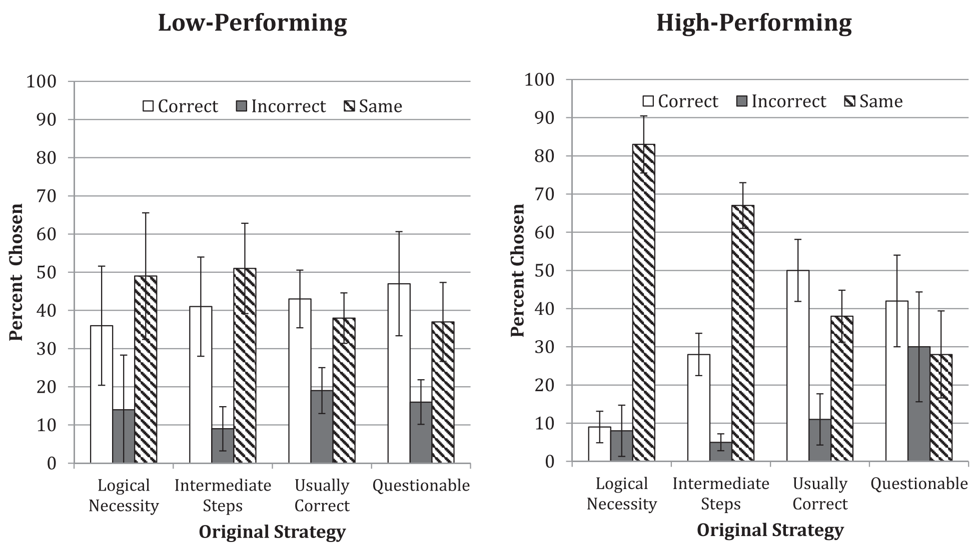

Figure 4. Percent of trials on which community college participants switched to a correct strategy, switched to an incorrect strategy, or stayed with their original strategy, given use of each of the four strategy types when the problem was originally presented (Experiment 3). Error bars are standard errors.

more often when their original strategy was questionable, but the slope was only marginally different from zero, $b = 6.11$, $F(1, 38) = 3.33$, $p = .076$. As with their other choices, low-performing participants' switches to an incorrect strategy were independent of their original strategy, $b = 1.45$, $F < 1$. There was no interaction, indicating that the slopes did not differ between the two groups, $F < 1$.

Equal denominators versus equal numerators. Given the greater reliance on equal denominator than equal numerator strategies in Experiments 1 and 2, we were interested in whether participants also more often switched to correct strategies that involved equal denominators. This proved to be the case. Participants switched to the equal denominator strategy on 64% ($SD = 38$) of Phase 2 equal denominator problems on which they had not chosen that strategy in Phase 1, but switched to the equal numerator strategy on only 41% ($SD = 35$) of equal numerator problems. Similarly, when participants did not originally use the multiply for common denominator strategy, they switched to it in Phase 2 on 43% ($SD = 43$) of the multiply for common denominator problems, but only switched to the equal numerator strategy on 19% ($SD = 28$) of the equal numerator trials. There were no significant differences in frequency of switches between high- and low-performing participants, all ts < 1.

Discussion

When participants were presented both correct and incorrect alternatives to their original strategy, they usually maintained their original choice. When they did switch, they usually switched to a correct alternative. Both high- and low-performing students showed this tendency.

The most interesting results came from examining the likelihood of switching among specific types of strategies. The high-performing community college students generally chose their original strategy when it was effective and switched to a correct alternative when the original strategy was questionable. This pattern of switching suggested that these students' use of questionable strategies was attributable to their failing to remember a better strategy, rather than to not understanding that their initial strategy was suboptimal.

In contrast, low-performing students rarely recognized that a correct alternative was superior to their original strategy, even when that original strategy was of questionable effectiveness. Thus, their difficulties with fraction magnitude comparison appeared to involve both a failure to remember correct procedures and a lack of conceptual understanding of correct procedures.

Just as students in all experiments more often used strategies involving equal denominators than equal numerators, those in Experiment 3 showed the same preference when given the opportunity to switch strategies. Possible reasons for this large difference in strategy use on logically equivalent comparison problems are discussed in the next section.

General Discussion

The trial-by-trial strategy assessments in this study shed light on adults' fraction magnitude comparison strategies, how these strategies vary with mathematical knowledge, and how limitations of memory and conceptual knowledge affect the choices. Results of Experiment 1 indicated that university students with high mathematical knowledge primarily rely on strategies that always lead to the correct answer if executed properly. In contrast, results of Experiments 2 and 3 indicated that community college students vary considerably in their fraction knowledge and strategy use. The high-performing community college students chose relatively effective strategies, though compared with the mathematically proficient university students in Experiment 1, they less often chose the most effective strategies. In contrast, low-performing community college students relied almost entirely on questionable strategies. Consistent with this analysis, magnitude comparisons of both the students at the highly selective university and the high-performing community college students were considerably more accurate than those of the low-performing community college students.

In Experiment 3, high-performing community college students showed some conceptual knowledge of fraction comparison strategies, disproportionately switching from less to more effective strategies when superior alternatives were presented. In contrast, strategy switches of low-performing students were unrelated to the quality of their original strategy, suggesting that they lacked understanding of alternative strategies for solving the problems.

Together, the three experiments allowed us to address the four issues about students' fraction magnitude comparison strategies that motivated the study: (a) the variability of strategy use, (b) the adaptiveness of strategy choices, (c) whether increases in speed and accuracy with additional mathematical knowledge could be explained by the four mechanisms described by overlapping waves theory, and (d) whether poor strategy choices are related to a failure to remember appropriate strategies or to a lack of understanding of the strategies.

Strategic Variability—Moving Beyond the Componential–Holistic Distinction

As expected, students at both a highly selective university and at a nonselective community college used numerous strategies to compare fraction magnitudes. In Experiment 1, students at the highly selective university used an average of 11 different strategies, with no student using fewer than six. In Experiments 2 and 3, more and less mathematically proficient students at a nonselective community college used between seven and 11 strategies. This variability of strategy use is far greater than revealed by previous studies of fraction magnitude comparison (Bonato et al., 2007; Meert et al., 2009, 2010; Schneider & Siegler, 2010). Rather than simply comparing numerators, denominators, or integrated magnitudes, participants visualized geometric representations of the fractions; reasoned that a fraction greater than half must be greater than one less than half; multiplied to obtain equal denominators or numerators; used the previously observed strategies of comparing numerator or denominator magnitudes; and employed many other approaches. This strategic variability indicates that fraction magnitude comparison is a much richer and more complex phenomenon than has emerged in previous depictions. The variability within individuals also raised the issue of how people choose which strategy to use on a given problem.

Adaptiveness of Strategy Choice

Students at the highly selective university were very adaptive in their strategy choices across the eight problem types—they effectively matched their strategies to the demands of each type of problem. On all types of problems, they most often used the strategy that was hypothesized to be most effective on that type of comparison.

Strategy choices of community college students were less adaptive. The high-performing community college students achieved fraction magnitude comparison accuracy similar to that of the mathematically sophisticated university students, but they were less likely to use strategies tailored to the specifics of the comparison problems, such as noting that the two fractions were on opposite sides of one half or that the same fraction had both a larger numerator and a smaller denominator. The low-performing community college students showed almost no recognition of differences among problem types. Other than the equal denominator and equal numerator problems, they never used a hypothesized strategy on more than 15% of the problems designed to elicit that strategy. Even for the equal denominator and equal numerator problems, they used the relevant strategy on less than half of the trials. Unlike in many domains in which low-knowledge individuals choose strategies adaptively (e.g., Kerkman & Siegler, 1993; Lemaire & Siegler, 1995), low-performing students rarely matched their fraction magnitude comparison strategy to the type of problem presented.

One interesting bias that emerged in the present data was that students were much more likely to base comparisons on equal denominators than equal numerators, despite the two strategies being logically equivalent for comparison problems. This generalization applied to students at both highly selective and nonselective institutions, to problems on which either the numerators or denominators were equal initially, and to problems for which equal numerators or denominators could be created by multiplying one of the fractions by N/N. This bias fits prior results from both Meert et al. (2009), who found that university students were slower and less accurate with equal numerator than equal denominator problems, and from Obersteiner et al. (2013), who found the same with expert mathematicians.

Several nonexclusive explanations of the difference in strategy choices on these formally equivalent problems seemed plausible. The difference might be attributable to students preferring to compare variables that vary positively with the outcome (greater numerator implies greater magnitude). Supporting this perspective, on problems for which one variable varies directly and another inversely with the outcome, children usually understand the impact of the variable that varies directly years before they understand the impact of the one that varies indirectly. For example, on shadow-projection problems, children understand the direct relation of the length of the object whose shadow is being cast to the shadow's length many years before they understand the inverse relation of distance from the light source to the shadow's length (Inhelder & Piaget, 1958; Siegler, 1981). Similarly, on time-judgment problems, children understand the direct relation between time of travel and distance traveled (holding speed constant) years before they understand the inverse relation between the speed at which an object travels and the temporal duration needed to travel a fixed distance (Piaget, 1969; Siegler & Richards, 1979).

Another plausible reason for the greater reliance on equal denominators than equal numerators is that equal denominators are necessary for adding and subtracting fractions, whereas equal numerators are irrelevant to fraction arithmetic. This probably leads to children having greater experience with fractions with equal denominators than ones with equal numerators. Such fraction addition and subtraction experience might also lead to children attending more closely to whether denominators are equal than to whether numerators are.

Effects of Mathematical Knowledge on Magnitude Comparison

As predicted, the mathematically proficient students at the highly selective university were both more accurate and faster when comparing fraction magnitudes than the less proficient students at the nonselective community college. These differences could be analyzed in terms of the four dimensions of change hypothesized by overlapping waves theory (Siegler, 1996): discovery of more advanced strategies, greater use of advanced strategies that were already known, improved strategy choice, and improved execution of known strategies.

Both students at the highly selective university and high-performing community college students used qualitatively different strategies than the low-performing community college students. A number of advanced strategies such as halves referencing, multiplying for a common numerator or denominator, and converting to a decimal were used almost exclusively by students with greater mathematical proficiency.

As shown in Figure 2, a number of differences were present in the type of strategies used by the three groups of participants. The largest difference was that the low-performing students relied most often on questionable strategies, whereas the high-performing community college students and university students relied most on intermediate steps strategies. In addition, the use of logical necessity strategies increased with fraction knowledge. Even useful strategies that were used by all three groups, such as noting that the numerators were equal and choosing the fraction with the smaller denominator, were used more often by the more knowledgeable students. As already discussed, there also were clear differences in strategy choices with increasing fraction knowledge. The low-performing community college students rarely matched their strategy to the type of problem presented, the high-performing community college students chose strategies more adaptively, and the university students were the most likely to match their strategy to the problem presented. Finally, there were differences in the ability to correctly execute known strategies. For example, high-performing community college students and students at the selective university were correct on more than 95% of the trials when they used intermediate steps strategies, versus 70% correct when low-performing community college students used them.

Sources of Use of Inferior Strategies

Both memory failures and lack of conceptual knowledge contributed to students' use of inferior strategies. For high-performing community college students, memory failures appeared to play the larger role. When presented their original strategy and both correct and incorrect alternatives, they switched away from their original strategy more often when that strategy was less likely than an alternative to produce a correct answer. This pattern of judgments suggested that these students realized when their original strategy was nonoptimal and recognized a better alternative. In contrast, the low-performing community college students were no more likely to switch to the superior strategy when their original strategy was questionable than when it was reliable. Rather than simply not remembering better strategies, these students failed to recognize when an alternative was superior to their original strategy. This pattern of judgments suggested that they did not understand why some alternative strategies were better than others.

Instructional Implications

Perhaps the most basic instructional implication of these findings is that many adults require additional instruction if they are to gain conceptual understanding of fractions. Both the present and previous magnitude comparison and number line estimation data suggest that a substantial percentage of community college students lack a clear understanding of the magnitudes implied by specific fractions. Without such understanding of fractions, many middle-income occupations are off limits (McCloskey, 2007; Murnane et al., 1995). Thus, improving fraction instruction clearly should be given high priority.

A second, more specific instructional implication is that teachers should place greater emphasis on how denominator size affects fraction magnitudes. Addition and subtraction of fractions is only possible with equal denominators, but middle school students' most common error in fraction addition and subtraction involves combining fractions with unequal denominators as if they were independent whole numbers (e.g., $2/3 + 3/4 = 5/7$; Hecht & Vagi, 2010; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). A central recommendation of the Common Core State Standards for teaching fractions (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) is that students should learn that nonunit fractions (fractions for which the numerator does not equal 1) are iterations of unit fractions (e.g., understanding $4/7$ as $1/7 + 1/7 + 1/7 + 1/7$). Such an approach, which is central to fraction instruction in East Asia (Son & Senk, 2010), might also help other students understand the crucial role of equal denominators in fraction addition and subtraction. Given the poor understanding of fractions shown by adults who attend community colleges, and given findings that interventions that focus on improving understanding of the role of denominators and fraction magnitudes also improve fraction arithmetic (Fuchs et al., 2013; Fuchs et al., 2014), the hypothesis certainly seems worth testing.

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