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Development of fraction concepts and procedures in U.S. and Chinese children

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ABSTRACT

We compared knowledge of fraction concepts and procedures among sixth and eighth graders in China and the United States. As anticipated, Chinese middle school children had higher knowledge of fraction concepts and procedures than U.S. children in the same grades, and the difference in procedural knowledge was much larger than the difference in conceptual knowledge. Of particular interest, national differences in knowledge of fraction concepts were fully mediated by differences in knowledge of fraction procedures, and differences between the knowledge of Chinese and U.S. children were most pronounced among the lowest achieving children within each country. Based on these and previous results, a theoretical model of the mutually facilitative interaction between conceptual and procedural knowledge of fractions is proposed and discussed.

Introduction

Fraction knowledge plays a vital role in children's mathematical development. Fractions (along with the closely related concepts of ratios and proportions) are ubiquitous in algebra, in more

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advanced mathematics, and in the sciences. Moreover, elementary school children's fraction knowledge uniquely predicts their mathematics achievement and their knowledge of algebra in high school above and beyond IQ, reading achievement, working memory, family education and income, and whole number arithmetic knowledge (Siegler et al., 2012). Furthermore, in both the United States and Great Britain, fraction knowledge and mathematics achievement continue to be highly correlated even when both are measured in high school ($rs > .80$).

Procedural and conceptual knowledge of fractions

An important distinction between different types of numerical knowledge, one that applies to all numbers, is that between procedural and conceptual knowledge (Hiebert & LeFevre, 1986). Procedural knowledge of fractions consists of fluency with the four fraction arithmetic operations: addition, subtraction, multiplication, and division. Conceptual knowledge in this area involves understanding the properties of fractions, including their magnitudes (e.g., 4/5 is greater than 1/2), principles relevant to fractions (e.g., an infinite number of fractions can be placed between any two other fractions), and notations for expressing fractions (e.g., 3/4 = 6/8 = .75).

Recent theoretical analyses and empirical research have focused on the influence of knowledge of fraction concepts on the learning of fraction procedures. For example, the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011) highlights the importance of understanding fraction magnitudes for acquiring procedural knowledge of fractions. Within this theory, children who do not understand fraction magnitudes are viewed as being at a significant disadvantage for learning fraction procedures because they cannot estimate solutions to arithmetic problems and, therefore, are unlikely to detect solutions that are implausible or to reject flawed procedures that produced such answers. Consistent with this analysis, knowledge of fraction concepts and knowledge of fraction procedures are moderately to highly correlated (Byrnes & Wasik, 1991; Hallett, Nunes, & Bryant, 2010; Hecht, 1998; Hecht, Close, & Santisi, 2003; Siegler & Pyke, 2013; Siegler et al., 2011). Stronger evidence for the hypothesized relation comes from two randomized control trials in which the intervention focused on increasing children's fraction magnitude knowledge. Relative to standard classroom instruction that emphasized the part–whole interpretation of fractions and fraction arithmetic, the intervention produced greater gains not only in fraction magnitude knowledge but also in fraction arithmetic, with the improvement in arithmetic being mediated by the increases in magnitude knowledge (Fuchs, Schumacher, et al., 2013; Fuchs et al., 2014).

Because evidence was already available that conceptual knowledge of fractions influences acquisition of fraction arithmetic procedures (e.g., Byrnes & Wasik, 1991; Hiebert & Wearne, 1986), in the current study we focused on whether the inverse relation is also present—whether knowledge of fraction procedures influences the development of knowledge of fraction concepts. Evidence for such bidirectional relations between conceptual and procedural knowledge has been found for other types of mathematics problems. For example, Rittle-Johnson, Siegler, and Alibali (2001) found that, for problems including decimals, conceptual knowledge predicted gains in procedural knowledge, which predicted subsequent gains in conceptual knowledge. Direct instruction on whole number procedures or concepts has been shown to increase the other type of knowledge (Rittle-Johnson & Alibali, 1999). In addition, practicing whole number procedures has been shown to increase knowledge of whole number concepts (Canobi, 2009) and also to stimulate discovery of novel whole number procedures that require more advanced conceptual understanding than previous procedures (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013; Siegler & Jenkins, 1989). The goal here was to determine whether similar bidirectional relations are present with common fractions, whose acquisition tends to be much more difficult than that of whole numbers or decimals (DeWolf, Grounds, Bassok, & Holyoak, 2014).

One explanation that has been proposed to account for the positive influence of improved conceptual knowledge on procedural competence is that greater conceptual understanding produces improved problem representations, which then have a positive influence on execution of procedures (Rittle-Johnson et al., 2001). One reason to hypothesize that improved procedural knowledge improves conceptual understanding is that skillful execution of fraction arithmetic procedures provides information regarding the magnitudes of the operands and answers in the fraction arithmetic problems. For example, many children understand the magnitudes of fractions less than 1 much better than they understand the magnitudes of fractions greater than 1 (Siegler & Pyke, 2013; Siegler et al., 2011). If children understand that the magnitude of 5/6 is a little less than 1, then learning to correctly execute the fraction addition procedure to solve problems such as 5/6 + 5/6 might help such children to understand that the answer (10/6 or 5/3) must be greater than 1 but less than 2. In contrast, children who frequently err on fraction arithmetic problems will not learn about magnitudes in this way because the magnitudes of the operands will not correspond to reasonable estimates of the answer. For example, the common error of first adding numerators and then independently adding denominators (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010) leads to the answer 5/6 + 5/6 = 10/12, which is equivalent to claiming that adding the two numbers has no effect on the answer's magnitude or to the equally wrong conclusion that 10/12 > 5/6. Such procedural errors seem likely to interfere with developing accurate representations of the magnitudes of answers to many fraction arithmetic problems.

To examine whether the relation between knowledge of fraction concepts and procedures is bidirectional, and in particular to provide evidence for a procedural-to-conceptual link, we examined knowledge of fraction concepts and procedures in children of different ages in China and the United States. These countries were chosen because in previous studies of young children's whole number knowledge, Chinese children were superior in both conceptual and procedural knowledge and the differences in procedural knowledge were especially large (Siegler & Mu, 2008; Stevenson et al., 1990). If this pattern generalizes to fractions, and if procedural knowledge influences acquisition of conceptual knowledge, national differences in knowledge of fraction procedures should mediate national differences in knowledge of fraction concepts.

An alternative hypothesis is that national differences in knowledge of fraction concepts and procedures arise independently. One argument for this alternative hypothesis is empirical. Despite the evidence described above that practicing whole number procedures can increase knowledge of whole number concepts, no similar demonstrations exist with fraction procedures and concepts. Given the high prevalence of incorrect fraction procedures among U.S. children (Ni & Zhou, 2005), even the potential frequency of such transfer is limited in that population. However, the absence of evidence for transfer from knowledge of fraction procedures to knowledge of fraction concepts is not evidence against this possibility; there is not much evidence one way or the other.

Another potential reason why knowledge of fraction procedures might not affect knowledge of fraction concepts is that differences in acquisition of knowledge of fraction concepts and procedures might be shaped by different variables. For example, Chinese children's greater conceptual knowledge might reflect their being instructed by teachers who more deeply understand the mathematics they are teaching (Ma, 1999), whereas Chinese children's greater procedural knowledge might reflect greater time spent on mathematics exercises and homework (e.g., Stevenson, Lee, & Stigler, 1986). To provide evidence relevant to the proposed link from procedural to conceptual knowledge, the current study addressed whether for fractions national differences in procedural knowledge mediate national differences in conceptual knowledge.

Hypotheses tested in the current study

This study addressed five hypotheses:

1. Chinese middle school children have greater knowledge of fraction procedures than their U.S. age/grade peers.
2. Chinese middle school children have greater knowledge of fraction concepts than their U.S. age/grade peers, but this difference is smaller than the difference in knowledge of fraction procedures.
3. Within both China and the United States, knowledge of fraction concepts, knowledge of fraction procedures, and overall mathematics achievement are moderately to highly correlated.
4. National differences in knowledge of fraction concepts are mediated by differences in knowledge of fraction procedures.
5. Chinese–U.S. differences in fraction knowledge are larger for low achieving students than for typically achieving students in each country.

Our first hypothesis was that Chinese middle school students have greater knowledge of fraction procedures than U.S. children in the same grade. Stevenson and colleagues (1990) and Cai (1995, 2000) found large differences between Chinese and U.S. children's whole number arithmetic fluency. Within the United States, whole number arithmetic skill in first grade predicts fraction arithmetic skill in middle school even after controlling for child IQ, working memory, race, and gender as well as parental income and education (Bailey, Siegler, & Geary, 2014). Together, these findings suggest that Chinese middle school children's fraction arithmetic fluency should be more advanced than that of their U.S. peers.

Our second hypothesis was that Chinese middle school children have greater knowledge of fraction concepts than same-grade U.S. children, but the difference is smaller than that in procedural knowledge. Previous studies have shown this pattern with whole numbers; Chinese children have greater knowledge of whole number concepts than their U.S. grade peers (Siegler & Mu, 2008; Stevenson et al., 1990; Zhou, Peverly, & Lin, 2005), but this difference tends to be smaller than differences in knowledge of whole number procedures (Cai, 1995, 2000). The same pattern was expected with fractions.

Our third hypothesis was that within both China and the United States, individual differences in knowledge of fraction concepts, knowledge of fraction procedures, and mathematics achievement are moderately to highly correlated. Prior studies of children in the United States have yielded such relations (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Byrnes & Wasik, 1991; Hallett et al., 2010; Hecht, 1998; Hecht et al., 2003; Siegler & Pyke, 2013; Siegler et al., 2011). Obtaining the same relations in China would provide evidence for the generality of these relations to populations with higher levels of mathematics achievement and different instruction (Yang & Cobb, 1995).

Our fourth hypothesis was that U.S.–Chinese differences in knowledge of fraction concepts are mediated by differences in knowledge of fraction procedures. As noted, this prediction is based on the logic that correct execution of fraction arithmetic procedures can help children to better understand the magnitudes of the fractions that are being combined arithmetically along with previous findings of transfer from knowledge of procedures to concepts with decimals and whole numbers. Note that this hypothesis in no way argues against conceptual knowledge influencing acquisition of procedural knowledge. Our view is that the relation is bidirectional, with conceptual and procedural knowledge interacting in a hand-over-hand process, whereby acquisition of each type of knowledge facilitates acquisition of the other (Rittle-Johnson & Schneider, in press; Rittle-Johnson & Siegler, 1998; Schneider, Rittle-Johnson, & Star, 2011).

Our fifth hypothesis was that national differences in knowledge of fraction concepts and procedures would be largest among relatively low achieving students in the two countries. East Asian teachers provide more thorough explanations of the conceptual basis of fraction arithmetic procedures than U.S. teachers (Ma, 1999; Moseley, Okamoto, & Ishida, 2007). Clear explanations probably matter most for low achieving children because those children are less likely to generate the rationales for themselves. Indirect evidence for this hypothesis comes from Siegler and Pyke's (2013) finding that the gap in knowledge of fraction procedures between low achieving and typically achieving U.S. children grew substantially between sixth and eighth grades because low achievers in eighth grade did only slightly better than low achievers in sixth grade. This pattern of development is often found for tasks on which many learners fail to move past a novice level of performance (Ackerman, 2007; Ackerman & Cianciolo, 2000). In contrast, when all or nearly all beginners learn to succeed on relatively simple tasks, this leads to a developmental pattern in which individual differences tend to *decrease* during skill development (Ackerman, 2007; Ackerman & Woltz, 1994). Thus, superior mathematics instruction in China was expected to provide all children there with a foundation for subsequent learning that many low achieving U.S. children lack, thereby increasing later gaps between children who are low achievers in the Chinese context and children who are low achievers in the U.S. context.

Method

Participants

The Chinese sample consisted of 44 sixth graders (mean age = 12.43 years, SD = 0.62, 52% female) and 39 eighth graders (mean age = 14.59 years, SD = 0.82, 54% female) from four middle-income public schools in Miyun County, Beijing. The U.S. sample was the one described by Siegler and colleagues (2011). Participants were 24 sixth graders (mean age = 11.69 years, *SD* = 0.44, 50% female, 88% Caucasian, 8% Asian, 4% biracial) and 24 eighth graders (mean age = 13.69 years, *SD* = 0.61, 50% female, 92% Caucasian, 4% Asian, 4% Hispanic) from two middle-income public school districts near Pittsburgh, Pennsylvania. U.S. participants' mathematics achievement was somewhat higher than the state average but representative of students at their schools; the percentages of children scoring at or above "proficient" on the state mathematics achievement test were 81% for participants, 74% for sixth and eighth graders in the state, and 79% for the schools from which the participants were sampled (http://www.portal.state.pa.us/portal/server.pt/community/school_assessments/7442/2008-2009_pssa_and_ayp_results/60028).

Chinese and U.S. participants encountered fraction concepts and procedures at similar times during their schooling. Chinese participants were introduced to the fraction magnitude concept in Grade 3, and U.S. participants were introduced to that concept in Grades 3 and 4. Chinese participants were introduced to basic fraction addition and subtraction with a common denominator in Grade 3, and U.S. participants were introduced to this material in Grade 4. Both Chinese and U.S. participants were introduced to fraction addition and subtraction with different denominators in Grade 5 and to fraction multiplication and division in Grade 6.

Fraction procedural knowledge task

Participants completed eight fraction arithmetic problems: two of each arithmetic operation, with one of the two having operands with equal denominators and the other having operands with unequal denominators. The items were 3/5 + 1/2, 3/5 + 2/5, 3/5 − 1/2, 3/5 − 2/5, 3/5 * 1/2, 3/5 * 2/5, 3/5 ÷ 1/2, and 3/5 ÷ 2/5. Items appeared one at a time on a computer screen. Participants were provided scratch paper to use while computing answers but entered the answers via a computer keyboard. Fractions equivalent to the correct answers were scored as correct even if they were not in simplest form.

Fraction conceptual knowledge tasks

0–1 Number line estimation

Participants viewed 10 number lines, one at a time, on a computer screen. Each line's left endpoint was labeled "0", and the right endpoint was labeled "1". On each trial, the fraction that participants were to estimate was presented above the middle of the number line. Each fraction was from a different tenth of the number line, and numerators and denominators were selected to limit the correlations between the fractions' magnitudes and those of their components. The fractions were 1/19, 1/7, 1/4, 3/8, 1/2, 4/7, 2/3, 7/9, 5/6, and 12/13. Participants responded by moving the cursor to the position of their desired estimate and clicking the mouse. Performance was measured as each participant's mean percentage absolute error (PAE), that is, the percentage deviation of each estimate from the correct location of the fraction on the number line.

0–5 Number line estimation

This task and the measure of performance on it were identical to those on the 0–1 number line estimation task except that the right endpoint of the number line was labeled "5" rather than "1" and the items to be estimated were 1/19, 4/7, 7/5, 13/9, 8/3, 11/4, 10/3, 7/2, 17/4, and 9/2.

Fraction magnitude comparison

Participants compared eight fractions' magnitudes with the fraction 3/5: 3/8, 5/8, 2/9, 4/5, 4/7, 5/9, 8/9, and 2/3. Stimuli from the magnitude comparison task were selected on the basis of their relation to the fraction 3/5. We oversampled the range of values close to the comparison fraction, 3/5, and included one fraction with the same numerator (3/8) and one with the same denominator (4/5). Each fraction appeared on the computer screen, one at a time, and participants were instructed to press the "a" key if the fraction on the screen was smaller than 3/5 and to press the "l" key if the fraction was larger than 3/5. Accuracy was each participant's percentage correct.

Achievement tests

To measure mathematics and reading achievement in the U.S. sample, we obtained participants' scores from the end of fifth and seventh grades (approximately half a year before the current study was conducted) on the mathematics and reading sections of the Pennsylvania System of School Assessment (PSSA), the standardized test used where the U.S. part of the study was conducted. The PSSA mathematics section examined a range of skills, including knowledge of whole number and fraction arithmetic; probability and statistics; interpretation of tables, graphs, and figures; pre-algebra; geometry; and series extrapolation. The PSSA reading section, which was used as a control in some analyses, assessed vocabulary and passage comprehension, interpretation, and analysis.

To measure mathematics and reading achievement in the Chinese sample, we obtained participants' scores on an achievement test developed by the Education Committee of Miyun County, Beijing. The mathematics section of the achievement test examined a range of skills, including arithmetic, geometry, and applied word problems. The reading section assessed pronunciation and writing of Chinese characters, vocabulary, reading comprehension, and composition.

Procedure

U.S. and Chinese participants were tested individually in a quiet room in their schools. All of the tasks were completed on a laptop computer during a single 30-min session. The order of tasks and the order of items within tasks were randomized; children completed all items within a task before beginning the next task. Children were allowed to use scratch paper for the procedural task but not for any of the tests of knowledge of fraction concepts.

Results

National and developmental differences in knowledge of fraction procedures

To examine whether Chinese children were more accurate than U.S. children on fraction arithmetic, we computed a two-way analysis of variance (ANOVA) with percentage correct as the dependent variable and with country, grade, and their interaction as predictors. As illustrated in Fig. 1, country and grade interacted, $F(1,127) = 10.96$, $p = .001$, $\eta_p^2 = .08$. In addition to the large main effect favoring Chinese students, the difference in accuracy between U.S. sixth and eighth graders (32% vs. 60%) was much larger than the difference between Chinese sixth and eighth graders (90% vs. 93%).

Because interactions often alter the interpretations of main effects, we report pairwise comparisons across country and grade combinations using p values obtained from Tukey's HSD (honestly significant difference) test. National differences were large, favoring Chinese children over U.S. children in sixth grade, Cohen's $d = 2.91$, $p < .001$, and in eighth grade, $d = 1.24$, $p < .001$. Chinese eighth graders were not significantly more accurate than Chinese sixth graders, $d = 0.17$, $p = .96$, likely because sixth graders were already close to ceiling level performance. In contrast, U.S. eighth graders were considerably more accurate than U.S. sixth graders, $d = 0.97$, $p < .001$, with both being far from ceiling level performance.

National and developmental differences in knowledge of fraction concepts

To examine whether Chinese sixth and eighth graders had greater knowledge of fraction concepts than their U.S. peers, we computed parallel two-way ANOVAs for the three measures of knowledge of fraction concepts. The country $*$ grade interaction did not reach significance in any of the models, all $Fs(1,127) < 1.40$, all $ps > .20$, all η_p^2 values <.02, indicating that age/grade difference in the two countries on these measures of conceptual knowledge did not differ. Therefore, this interaction was dropped from each model.

In the model predicting 0–5 number line PAE, there was a large effect of country, $F(1,128) = 29.47$, $p < .001$, $\eta_p^2 = .19$, indicating that Chinese middle school students estimated more accurately than U.S.

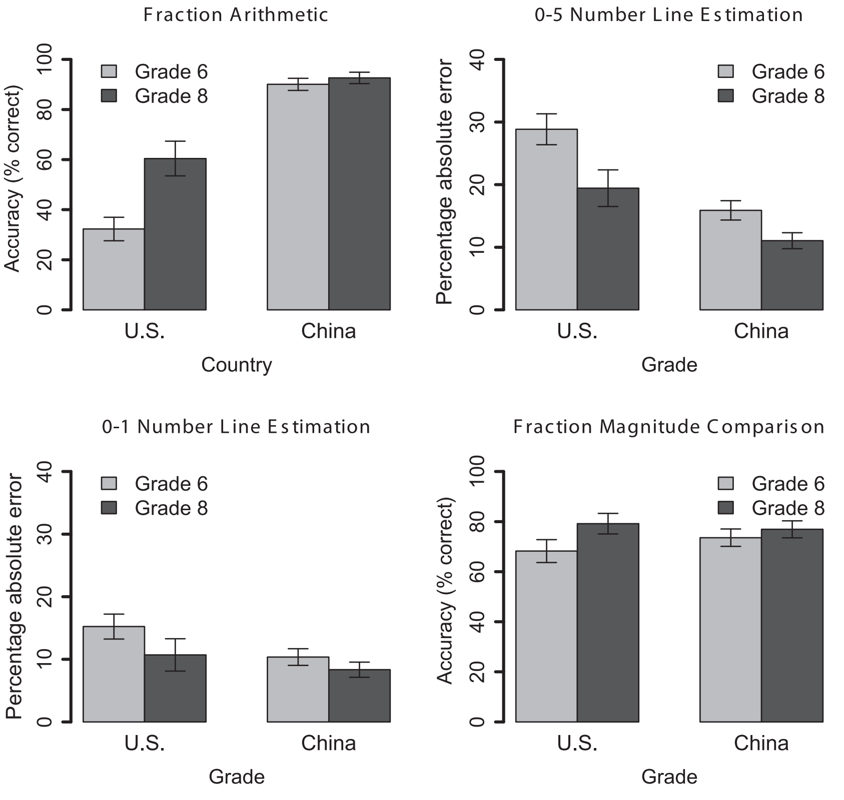

Fig. 1. National and developmental differences in fraction knowledge. Means and standard errors are displayed.

middle school students (PAEs = 14% vs. 24%, d = 0.88). (Lower PAEs indicate smaller error and, therefore, greater accuracy.) There was also an effect of grade, $F(1,128) = 11.74$, $p < .001$, $\eta_p^2 = .08$, with children in both countries estimating more accurately in eighth grade than in sixth grade (PAEs = 14% vs. 20%, d = 0.51).

In the model predicting 0–1 number line PAE, there was an effect of country, $F(1,128) = 4.49$, $p = .04$, $\eta_p^2 = .03$, indicating that Chinese middle school children were more accurate than their U.S. peers (PAEs = 9% vs. 13%, d = 0.37), but estimation accuracy did not differ between eighth and sixth graders (PAEs = 9% vs. 12%), $F(1,128) = 3.15$, $p = .08$, $\eta_p^2 = .02$.

In the model predicting fraction magnitude comparison accuracy, there was no interaction and no effect of country, $F(1,128) = 0.17$, $p = .68$, $\eta_p^2 = .001$, or grade, $F(1,128) = 2.57$, $p = .11$, $\eta_p^2 = .02$. Chinese middle school children were no more accurate than U.S. middle school children (75% vs. 74%, d = 0.07), nor were eighth graders any more accurate than sixth graders (78% vs. 72%, d = 0.28).

Mediation analyses

Chinese children reached a very high degree of mastery of fraction arithmetic by sixth grade. This finding, together with previous findings that procedural knowledge can lead to gains in conceptual knowledge of decimals (Rittle-Johnson et al., 2001), suggested that differences between Chinese and U.S. middle school students in knowledge of fraction concepts might be mediated by differences in their knowledge of fraction arithmetic procedures. Mediation analyses, which are used to test whether a proposed causal pathway passes through an intermediate variable, were conducted using the mediation package in R (Tingley, Yamamoto, Keele, & Imai, 2013). (For a detailed description of mediation analysis, see MacKinnon, 2008.) The 95% confidence intervals for the total, indirect, and direct effects were calculated using a nonparametric bootstrap with 1000 bootstrap iterations.

In the mediation analysis that we used to test this hypothesis, we first identified all task–grade combinations on which there was a statistically significant national difference. Chinese sixth graders estimated more accurately than U.S. sixth graders on both the 0–1 and 0–5 number line tasks, $t(66) = 2.09$, $p = .04$, $d = 0.52$, and $t(66) = 4.67$, $p < .001$, $d = 1.15$, respectively. Chinese eighth graders estimated more accurately than U.S. eighth graders on the 0–5 number line task, $t(61) = 2.99$, $p = .004$, $d = 0.72$.

Next, we tested whether these national differences in fraction number line estimation were mediated by fraction arithmetic knowledge in the same grade. In all three cases, the Chinese–U.S. difference in number line estimation was fully mediated by fraction arithmetic knowledge (Table 1, first three rows). Country exercised an indirect effect on sixth graders' 0–1 number line estimation accuracy through their fraction arithmetic accuracy, $\beta = .35$, 95% confidence interval $(CI) = (.03,.76)$, but no direct effect, $\beta = -.10$, 95% $CI = (-.53, .28)$. Similarly, country exercised an indirect effect on sixth graders' 0–5 number line estimation accuracy through their fraction arithmetic accuracy, $\beta = .48$, 95% $CI = (.12,.99)$, but no direct effect, $\beta = .02$, 95% $CI = (-.58, .49)$. Finally, country exercised an indirect effect on eighth graders' 0–5 number line estimation accuracy through their fraction arithmetic accuracy, $\beta = .41$, 95% $CI = (.18,.70)$, but no direct effect, $\beta = -.05$, 95% $CI = (-.21, .10)$. The lack of direct country effects in all three cases, together with the indirect effects involving fraction arithmetic in all three, indicated that fraction arithmetic proficiency fully mediated the national differences in number line estimation.

Results of these mediation analyses raised the question of whether the opposite pattern of mediation would also be present: Would national differences in conceptual knowledge fully mediate national differences in procedural knowledge, in which case no conclusion could be drawn from these mediation analyses? In two of these three mediation analyses, knowledge of fraction concepts partially mediated national differences in knowledge of fraction procedures (Table 1, bottom three rows). In the third case, the confidence interval for the mediating effect of conceptual knowledge included

Table 1
Mediation tests.

Initial variable	Mediator	Outcome	Path a (*SE*)	Path b (*SE*)	Total effect (95% *CI*)	Indirect effect (95% *CI*)	Direct effect (95% *CI*)	Proportion via mediation
Country	Sixth grade FA	Sixth grade NL 0–1	.83** (.07)	.42† (.21)	.25 (.02, .50)	.35 (.03, .76)	−.10 (−.53, .28)	>1.00
Country	Sixth grade FA	Sixth grade NL 0–5	.83** (.07)	.58** (.18)	.50 (.28, .72)	.48 (.12, .99)	.02 (−.58, .49)	.97
Country	Eighth grade FA	Eighth grade NL 0–5	.56** (.11)	.74** (.11)	.36 (.10, .65)	.41 (.18, .70)	−.05 (−.21, .10)	>1.00
Country	Sixth grade NL 0–1	Sixth grade FA	.25* (.12)	.14† (.07)	.83 (.67, .97)	.03 (.00, .11)	.80 (.64, .95)	.04
Country	Sixth grade NL 0–5	Sixth grade FA	.50** (.11)	.24** (.07)	.83 (.69, .97)	.12 (.03, .21)	.71 (.57, .87)	.14
Country	Eighth grade NL 0–5	Eighth grade FA	.36** (.12)	.58** (.09)	.56 (.31, .82)	.21 (.05, .40)	.35 (.14, .58)	.37

Note. The Country variable is coded as United States = 0, China = 1. NL 0–1, 0–1 number line estimation accuracy (−1 ∗ PAE); NL 0–5, 0–5 number line estimation accuracy (−1 ∗ PAE); FA, fraction arithmetic accuracy. The 95% confidence intervals were calculated using a nonparametric bootstrap with 1000 bootstrap iterations.
† $p = .0503$.
* $p < .05$.
** $p < .01$.

zero. The proportions of the national differences in procedural knowledge arising via mediation from conceptual knowledge (.04 to .37) also were much lower than the proportions of the national differences in conceptual knowledge arising through procedural knowledge (.97 to more than 1.00).

Moreover, the partial mediation of national differences in procedural knowledge by conceptual knowledge would be expected even if national differences in procedural knowledge entirely caused national differences in conceptual knowledge. The reason is that smaller but still significant mediation effects are often observed when the mediator and outcome variables are reversed even if the outcome does not cause the mediator (for a discussion of this issue, see Kenny, Kashy, & Bolger, 1998, p. 262). Indeed, our results are consistent with this pattern. The indirect effects of country on conceptual knowledge via procedural knowledge, for each measure of conceptual knowledge on which countries differed (sixth graders' 0–1 number line estimation and sixth and eighth graders' 0–5 number line estimation, βs = .35, .48, and .41), were noticeably larger than the indirect effects of country on procedural knowledge via conceptual knowledge (βs = .03, .12, and .21). All of the former effects fell outside of the 95% confidence intervals for the latter effects.

Most important, our results are informative regardless of the true effect of conceptual knowledge on national differences in procedural knowledge. Specifically, if national differences in procedural knowledge did not influence national differences in conceptual knowledge, then why would national differences in procedural fraction knowledge fully mediate national differences in conceptual fraction knowledge but not the reverse?

National and developmental differences in low achieving children's fraction knowledge

To test the hypothesis that differences between U.S. and Chinese students' fraction knowledge are strongest among children with low mathematical knowledge relative to peers in their country, we divided performance on each task into that of children whose scores were in the lowest one third of the distribution for their country (labeled "low achieving children") and those whose scores were in the upper two thirds (labeled "typically achieving children"). This division follows those in previous studies of children with low mathematics knowledge, including Hanich, Jordan, Kaplan, and Dick (2001), Jordan, Hanich, and Kaplan (2003), and Siegler and Pyke (2013). We then computed parallel country by achievement level ANOVAs, separately for each grade, for each measure of fraction knowledge. A significant country $*$ achievement level interaction indicates that the size of the national difference depends on children's achievement level.

The model of sixth graders' fraction arithmetic accuracy yielded the predicted country $*$ achievement level interaction, $F(1,64) = 4.35$, $p = .04$, $\eta_p^2 = .06$ (Fig. 2). Typically achieving Chinese sixth graders' fraction arithmetic was much more accurate than that of typically achieving U.S. sixth graders (100% vs. 52% correct), but the difference between low achieving Chinese sixth graders and low achieving U.S. sixth graders was even larger (76% vs. 15%).

The model predicting eighth graders' fraction arithmetic accuracy yielded a parallel but larger country $*$ achievement level interaction, $F(1,59) = 26.48$, $p < .001$, $\eta_p^2 = .31$. The difference between typically achieving Chinese and U.S. eighth graders' fraction arithmetic accuracy was not trivial (100% vs. 80%), but the difference between low achieving Chinese and U.S. eighth graders' fraction arithmetic accuracy was far larger (78% vs. 20%). Ceiling effects clearly limited the possible size of the difference among typically achieving children in the two countries, but the absolute difference between the arithmetic accuracy of low achieving children in the two countries (76% vs. 15% for sixth graders, 78% vs. 20% for eighth graders) was striking.

The model predicting eighth graders' 0–5 fraction number line PAE also yielded the predicted country $*$ achievement level interaction, $F(1,59) = 12.40$, $p < .001$, $\eta_p^2 = .17$. Typically achieving Chinese and U.S. eighth graders showed much more similar PAEs (7% vs. 11%) than low achieving Chinese and U.S. eighth graders (20% vs. 36%). In contrast, the model predicting sixth graders' 0–5 fraction number line PAE did not yield a country $*$ achievement level interaction, $F(1,64) = 0.07$, $p = .79$, $\eta_p^2 = .001$. The estimates of typically achieving Chinese sixth graders showed lower PAEs than those of typically achieving U.S. sixth graders (10% vs. 23%), as did the estimates of low achieving Chinese sixth graders compared with low achieving U.S. sixth graders (27% vs. 41%). Thus, the national difference in 0–5

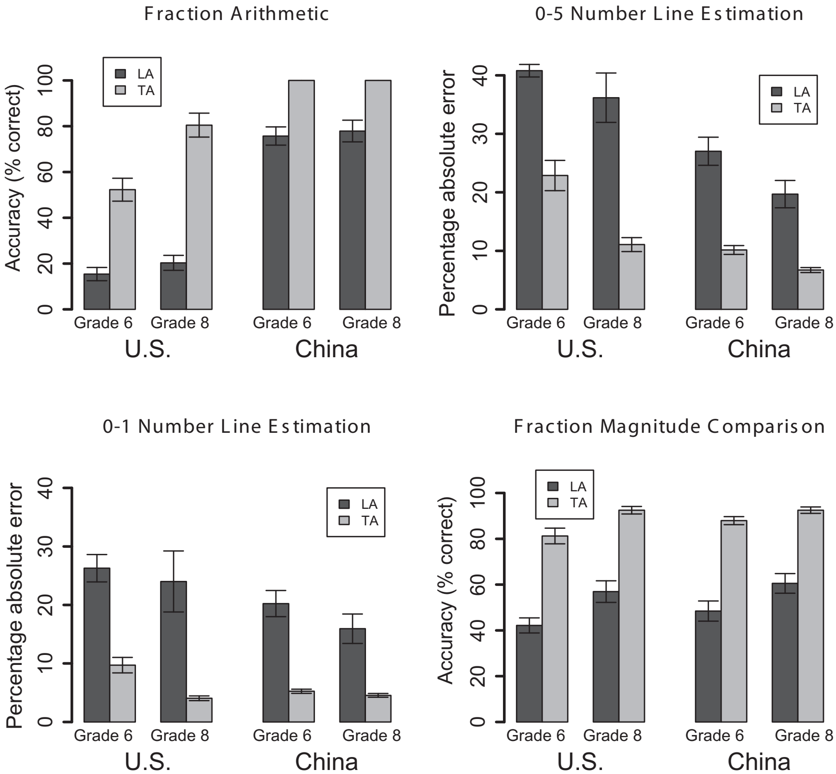

Fig. 2. Achievement level differences in fraction knowledge by country and grade. LA, low achieving; TA, typically achieving. Means and standard errors are displayed.

number line estimation accuracy was larger among low achieving children than among typically achieving children, but only for eighth graders.

A parallel model predicting eighth graders' 0–1 fraction number line PAE yielded the predicted country $*$ achievement level interaction, $F(1,59) = 5.47$, $p = .02$, $\eta_p^2 = .08$. Typically achieving Chinese and U.S. eighth graders showed highly similar 0–1 number line PAEs (5% vs. 4%), but low achieving Chinese eighth graders estimated considerably more accurately than their low achieving U.S. peers (16% vs. 24%). However, sixth graders' 0–1 fraction number line PAE did not yield a country $*$ achievement level interaction, $F(1,59) = 0.30$, $p = .59$, $\eta_p^2 = .005$. The estimates of typically achieving Chinese sixth graders showed lower PAEs than those of typically achieving U.S. sixth graders (5% vs. 10%), as did the estimates of low achieving Chinese sixth graders compared with low achieving U.S. sixth graders (20% vs. 26%). In summary, on both 0–1 and 0–5 number line estimation, the predicted larger difference between low achieving Chinese and U.S. students was present for eighth graders but not for sixth graders.

Parallel models predicting sixth and eighth graders' fraction magnitude comparison accuracy yielded neither main effects nor country $*$ achievement level interactions (Fs < 1, ps > .50, η_p^2 values < .01).

Consistency of individual differences

Relations of individual differences in performance on different tasks showed both similarities and differences across the two countries and grades (Table 2). One similarity was that fraction knowledge was consistently related to mathematics achievement in both countries and grades. Among sixth

Table 2
Correlations among tasks by country and grade.

	NL 0–1	NL 0–5	FMC	FA	Math	Reading
Sixth graders						
NL 0–1		.56***	−.48*	−.56**	−.66***	−.55**
NL 0–5	.34*		−.33	−.56**	−.54**	−.52*
FMC	.00	−.32*		.53**	.60**	.59**
FA	.03	−.21	.16		.48*	.64**
Math	−.35*	−.59***	.40**	.08		.68***
Reading	−.11	−.26	.10	.00	.52***	
Eighth graders						
NL 0–1		.61**	−.70***	−.64***	−.63**	−.62**
NL 0–5	.52***		−.67***	−.70***	−.86***	−.77***
FMC	−.06	−.03		.64***	.62**	.70***
FA	−.38*	−.55***	.16		.78***	.78***
Math	−.60***	−.79***	.24	.53***		.84***
Reading	−.58***	−.58***	.33*	.37*	.70***	

Note. FMC, fraction magnitude comparison accuracy; NL 0–1, 0–1 number line estimation PAE; NL 0–5, 0–5 number line estimation PAE; FA, fraction arithmetic accuracy; Math, mathematics achievement; Reading, reading achievement. Correlations for U.S. participants are above the diagonal; correlations for Chinese participants are below it.
* $p < .05$.
** $p < .01$.
*** $p < .001$.

graders, all measures of fraction knowledge were correlated with mathematics achievement except for Chinese children's fraction arithmetic accuracy. Among eighth graders, all measures of fraction knowledge were correlated with mathematics achievement except for Chinese children's fraction magnitude comparison accuracy.

Another similarity was that in both countries correlations between measures of fraction knowledge and mathematics achievement tended to be higher among eighth graders (United States: $|r|$ values ranged from .62 to .86; China: $|r|$ values ranged from .24 to .79) than among sixth graders (United States: $|r|$ values ranged from .48 to .66; China: $|r|$ values ranged from .08 to .59).

There also were differences between the patterns of correlations in the two countries. In particular, as shown in Table 2, correlations between different measures of fraction knowledge, as well as between fraction knowledge and reading achievement, were generally higher in the U.S. sample.

Discussion

The current findings were consistent with four of our five hypotheses. Consistent with the first hypothesis, Chinese middle school children had greater knowledge of fraction procedures than their U.S. peers. Consistent with the second hypothesis, Chinese middle school children had greater knowledge of fraction concepts than their U.S. peers, and these differences were smaller than the national difference in knowledge of fraction procedures. Consistent with the fourth hypothesis, all observed national differences in knowledge of fraction concepts were fully mediated by differences in knowledge of fraction procedures. Consistent with the fifth hypothesis, national differences were largest among low achieving children, with the national differences among low achieving children being greater in eighth grade than in sixth grade.

The one hypothesis that was disconfirmed by the current findings was that the correlations among knowledge of fraction concepts, knowledge of fraction procedures, and mathematics achievement would be high as high in China as in the United States. In part, this was attributable to ceiling effects on Chinese students' knowledge of fraction procedures; absolute majorities of both sixth and eighth grade Chinese children were literally at ceiling, and mean accuracy exceeded 90%. However, ceiling effects cannot explain the lower correlations in the Chinese sample among the three tasks measuring knowledge of fraction concepts or between them and the mathematics and reading achievement tests. The lower correlations in the Chinese sample also cannot be explained by the correlations in the U.S. sample not being replicable; comparable correlations among U.S. sixth and eighth graders have been observed by Siegler and Pyke (2013) and Bailey and colleagues (2014). Thus, beyond the ceiling effects on procedural knowledge, the reasons for the lower correlations among Chinese children remain to be identified.

Limitations

The current study had several limitations that should be addressed in future research. Most important, the cross-sectional design of the current study did not allow us to directly test the bidirectional explanation of national differences in children's fraction knowledge. That conclusion was based on combining previous evidence for the role of conceptual knowledge in acquiring procedural knowledge with the current evidence of procedural knowledge mediating national differences in conceptual knowledge. Longitudinal data will be required to test whether national differences in one type of knowledge predict gains in the other type of knowledge. These data would provide better evidence of causal mediation than the cross-sectional data presented in the current study. However, our cross-sectional findings argued against the alternative hypothesis that national differences in procedural knowledge were completely caused by national differences in conceptual knowledge and also against the alternative hypothesis that these differences arose completely independently of each other. If either of these hypotheses accounted for national differences in children's knowledge of fraction procedures and concepts, then national differences in knowledge of fraction procedures would be highly unlikely to fully mediate knowledge of fraction concepts for every combination of conceptual task and grade on which Chinese children outperformed U.S. children.

In addition, the Chinese participants in the current sample were older than their same-grade U.S. peers. This age difference could explain some of the national differences in fraction knowledge. However, age differences cannot account for the differences in the sizes of national differences in knowledge of fraction procedures versus concepts. Indeed, a substantial difference in knowledge of fraction procedures remained even if U.S. eighth graders were compared with younger Chinese sixth graders (Fig. 1).

The current study's sample had two limitations. First, the sample was large enough only to detect moderate to large national differences. For example, national differences too small to be detected with the current sample size might exist on the fraction magnitude comparison task. Larger longitudinal studies of national differences in fraction development will be required to capture nuances in the developmental process underlying fraction development. Second, the degree to which the Chinese and U.S. samples were nationally representative is unclear. Future studies would benefit from nationally representative samples.

Finally, our measures were not an exhaustive battery for assessing children's fraction knowledge. Assessing other facets of children's knowledge of concepts and procedures in future cross-national comparisons, including the strategies children use on these tasks, may help to clarify how patterns of national differences emerge.

In general, however, the current results were consistent with the perspective on which the study was based—that acquisition of knowledge of fraction concepts and procedures is a bidirectional hand-over-hand process, with each type of knowledge facilitating acquisition of the other. Previous findings indicated that knowledge of fraction concepts can influence the acquisition of knowledge of fraction procedures (Fuchs, Schumacher, et al., 2013; Hecht & Vagi, 2010; Hiebert & Wearne, 1986; Mack, 1990). These previous findings, together with the current finding that knowledge of fraction procedures fully mediated national differences in knowledge of fraction concepts, indicated that the relation is likely to be bidirectional. Similar findings from studies of decimals (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson et al., 2001) and equation solving (Schneider et al., 2011) indicate that this relation is not unique; indeed, we suspect that such bidirectional relations between conceptual and procedural knowledge are the rule, rather than the exception, in mathematics learning and quite likely in other domains as well (for examples, see Rittle-Johnson & Siegler, 1998).

Toward a model of children's fraction development

Fig. 3 outlines a model of fraction development that emphasizes mutually facilitative interactions between conceptual and procedural knowledge. Understanding of fraction magnitudes and

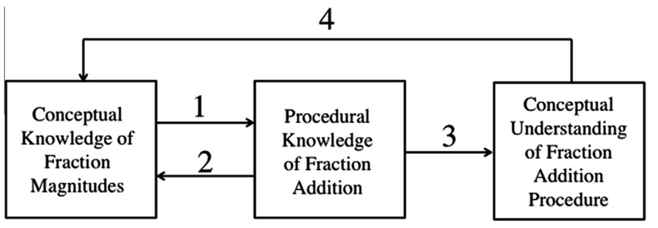

Fig. 3. Model of mutually facilitative interactions between conceptual and procedural knowledge of fractions.

understanding of why fraction arithmetic procedures work are the types of conceptual knowledge considered in the model; fraction addition is the procedure being considered. The model is admittedly speculative and contains several untested features, such as those involving understanding of how procedures work, but it illustrates how understanding of fractions might develop. Furthermore, the model generates several testable predictions that can be evaluated in future studies.

The developmental progression begins with learning the type of conceptual knowledge examined in the current study—magnitude knowledge of fractions from 0 to 1. Such knowledge often emerges through instruction and practice that helps children to map numerically expressed fractions (N_1/N_2) onto number lines, rectangles, and (especially) circles, for example, the omnipresent circular pizza representation.

Once acquired, such knowledge of fraction magnitudes could facilitate learning of fraction arithmetic procedures (Process 1 in Fig. 3). This could occur by allowing learners to reject arithmetic procedures that produce implausible answers. Such an application of magnitude knowledge to learning arithmetic procedures can be illustrated on the problem 1/2 + 1/2. A child who understood fraction magnitudes in the 0–1 range could reject one common incorrect fraction addition procedure—independently adding numerators and denominators—because that approach would yield an answer (2/4) that equals each addend, equivalent to saying that 1/2 + 1/2 = 1/2. The combination of the fraction magnitude knowledge and basic understanding of addition (knowing that adding positive numbers yields a sum greater than either addend) would indicate that this could not be correct. In contrast, the correct addition procedure would yield the answer "1", which would be consistent with the answer that would be expected on the basis of the child's magnitude knowledge. Strengthening procedures that yield correct answers, and weakening procedures that yield incorrect answers, are pervasive mechanisms in both symbolic and nonsymbolic models of learning (Anderson, 2005; Elman, 2005).

Knowledge of the fraction addition procedure, in turn, could increase magnitude knowledge in the 0–1 range by yielding answers whose magnitudes were consistent with those yielded by accurate representations of the individual addends (Process 2 in Fig. 3). In the 1/2 + 1/2 example, imagining the magnitude represented by 1/2 and correctly executing the procedure could strengthen a child's confidence that 2/2 = 1.

Knowledge of the fraction addition procedure also seems to be essential for acquiring a different type of knowledge of fraction concepts, one that we did not measure in the current study and that has not been given significant empirical attention—knowledge of how and why the addition procedure works (Process 3 in Fig. 3). Without knowing the procedure, there is nothing to explain. Once the addition procedure is learned, automatizing its execution would free cognitive resources for analyzing why the procedure yields reasonable answers and works as it does. This might lead to the recognition (conscious or unconscious) that the subgoal in the addition procedure of obtaining common denominators is crucial because in a problem such as 1/2 + 1/3, adding a number of halves to a number of thirds does not yield a number of halves, a number of thirds, or a number of any other unit. In contrast, converting both fractions to the common denominator of sixths yields a specific number of sixths (three of the sixths + two of the sixths = five of the sixths or 5/6).

Knowledge of the fraction addition procedure could also increase knowledge of fraction magnitudes beyond the 0–1 range (Process 4 in Fig. 3). For example, if a child knew fraction magnitudes in the 0–1 range but not magnitudes greater than 1, which is the case for many students (Siegler & Pyke, 2013; Siegler et al., 2011), then correctly executing the fraction addition procedure to obtain 2/3 + 3/4 = 17/12 would help to convey that 17/12 was around 1 – 1/2, correctly executing the procedure for 2 – 2/3 + 2 – 3/4 would indicate that the answer was somewhere around 5 – 1/2, and so forth.

The model outlined in Fig. 3, together with the current finding that differences between Chinese and U.S. children's fraction knowledge were largest among the lowest achieving children, raises an intriguing question for future research. The most straightforward explanation for Chinese children's superior procedural knowledge is greater practice with the procedures. East Asian mathematics instruction emphasizes practicing procedures to a greater extent than U.S. mathematics instruction (Stevenson & Stigler, 1992; Stigler, Chalip, & Miller, 1986), which likely contributes to the greater procedural fluency observed in Chinese children (Royer, Tronsky, Chan, Jackson, & Marchant, 1999). Moreover, extensive speeded practice has been shown to especially enhance low performing U.S. students' arithmetic learning (Fuchs, Geary, et al., 2013).

Fig. 3 suggests an additional (nonexclusive) explanation of this outcome. Chinese children's greater knowledge of fraction magnitudes contributes to the cascade of learning events described in the figure. That is, low achieving Chinese children's greater knowledge of fraction magnitudes, relative to their low achieving U.S. peers, could help the Chinese children to learn fraction arithmetic procedures more effectively, which in turn could facilitate acquisition of further conceptual knowledge of magnitudes and of the conceptual basis of fraction procedures. The widening gap between low achieving Chinese and U.S. children's fraction knowledge between sixth and eighth grades in the current study is consistent with this type of cascade. Following Chinese and U.S. children longitudinally, and assessing both conceptual and procedural aspects of their fraction knowledge at frequent intervals, could enrich our understanding of the interplay of conceptual and procedural knowledge in fraction learning. If the model proves to be useful in accounting for development of fraction knowledge, the basic framework could be extended to accounting for development in other numerical domains as well.

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